【学術論文】（査読あり）
京都工芸繊維大学学術報告書　第15巻（令和5年3月）
BULLETIN OF KYOTO INSTITUTE OF TECHNOLOGY　15 (Mar. 2023)

A Method for Direct Solution of the Dirac Equation

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（2021年10月4日原稿受理　2022年3月18日採用決定）

SUMMARY

The author has developed a method to find the direct solution of the Dirac equation. To solve the problem, first we determined the most general form of the successive Lorentz transformations. Then, we have found their corresponding representation matrix $S(\Lambda)$ in the representation space of the Dirac spinors. The matrix $S(\Lambda)$ is used for the purpose of both diagonalizing the Dirac operators and determining the Dirac spinors that give the solution of the Dirac equation.

The present method also helps study the constitution of Dirac spinors and Dirac operators. Furthermore, we can make the most of the method to address the topics of matrix algebra such as the polar decomposition and projection operator.

Key Words: Dirac equation, Dirac operator, Dirac spinor, Lorentz transformation, Polar decomposition, Projection operator.

1 Introduction

The Dirac equation has been long investigated as one of the basic equations of quantum mechanics, especially in the area of quantum field theory.[1)] The Dirac equation is described by

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0 \ \ (\mu = 0, 1, 2, 3), \tag{1}$$

where i is the imaginary unit; m is the mass of an electron; γ^{μ} are (4, 4) matrices called gamma matrices;[1)] ∂_{μ} is an abbreviation of $\partial/\partial x^{\mu}$; x is also an abbreviation of the space-time coordinates in the Minkowski space. The space-time coordinate of x in Eq. (1) is expressed as

$$x \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{2}$$

where x^0 denotes the time coordinate; x^1, x^2, and x^3 are the space coordinates. The function $\psi(x)$ is represented by a (4, 1) matrix (i.e., column vector) and called a Dirac spinor.[2] The representation of gamma matrices depends on how the Pauli matrices are defined. In this article, we adopt the Wigner's representation[3] for them. We list the gamma matrices and the Pauli spin matrices as follows:

$$\gamma^0 = \begin{pmatrix} E & \mathbf{0} \\ \mathbf{0} & -E \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} \mathbf{0} & \sigma_k \\ -\sigma_k & \mathbf{0} \end{pmatrix} \ (k = 1, 2, 3), \tag{3}$$

where each entry represents (2, 2) matrices with E and $\mathbf{0}$ being the identity matrix and zero matrix, respectively. Individual Pauli matrices $\sigma_k \ (k = 1, 2, 3)$ are given by[3]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

The two linearly-independent plane-wave solutions of $\psi(x)$ are expressed as

$$\phi(x) = u(\boldsymbol{p}, h) \, e^{-ipx}, \tag{5}$$

$$\chi(x) = v(\boldsymbol{p}, h) \, e^{ipx}, \tag{6}$$

where h denotes the helicity; $\boldsymbol{p}$ and p are momentum and four-momentum, respectively, with p defined as $p \equiv \begin{pmatrix} p^0 \\ \boldsymbol{p} \end{pmatrix}$ with p^0 (> 0).

Notice that the (4, 1) matrices $u(\boldsymbol{p}, h)$ and $v(\boldsymbol{p}, h)$ behave as a constant with respect to the space-time coordinate x. In Eqs. (5) and (6), $\phi(x)$ and $\chi(x)$ are referred to as the positive-energy solution and negative-energy solution, respectively. The energy of a particle (p^0) with its rest mass m is given by

$$p^0 = \sqrt{\boldsymbol{p}^2 + m^2}. \tag{7}$$

Substituting Eqs. (5) and (6) for Eq. (1), we obtain

$$(p_\mu \gamma^\mu - m) u(\boldsymbol{p}, h) = 0, \tag{8}$$

$$(-p_\mu \gamma^\mu - m) v(\boldsymbol{p}, h) = 0. \tag{9}$$

The full matrix representations for Eqs. (8) and (9) are given by

$$\begin{pmatrix} p^0 - m & 0 & p^3 & -p^1 - ip^2 \\ 0 & p^0 - m & -p^1 + ip^2 & -p^3 \\ -p^3 & p^1 + ip^2 & -p^0 - m & 0 \\ p^1 - ip^2 & p^3 & 0 & -p^0 - m \end{pmatrix} u(\boldsymbol{p}, h) = 0 \tag{10}$$

and

$$\begin{pmatrix} -p^0 - m & 0 & -p^3 & p^1 + ip^2 \\ 0 & -p^0 - m & p^1 - ip^2 & p^3 \\ p^3 & -p^1 - ip^2 & p^0 - m & 0 \\ -p^1 + ip^2 & -p^3 & 0 & p^0 - m \end{pmatrix} v(\boldsymbol{p}, h) = 0, \tag{11}$$

respectively. In this article, we call the (4, 4) matrices of Eqs. (10) and (11) Dirac operators.[4] Although the word "Dirac operator" is normally used for differential operator represented in Eq. (1), we use this word for (4, 4) matrices that are derived from the differential operators. For later use, we define the (4, 4) matrix operators of Eqs. (10) and (11) as

$$\mathfrak{G} \equiv \begin{pmatrix} p^0 - m & 0 & p^3 & -p^1 - ip^2 \\ 0 & p^0 - m & -p^1 + ip^2 & -p^3 \\ -p^3 & p^1 + ip^2 & -p^0 - m & 0 \\ p^1 - ip^2 & p^3 & 0 & -p^0 - m \end{pmatrix}, \tag{12}$$

$$\tilde{\mathfrak{G}} \equiv \begin{pmatrix} -p^0 - m & 0 & -p^3 & p^1 + ip^2 \\ 0 & -p^0 - m & p^1 - ip^2 & p^3 \\ p^3 & -p^1 - ip^2 & p^0 - m & 0 \\ -p^1 + ip^2 & -p^3 & 0 & p^0 - m \end{pmatrix}. \tag{13}$$

The properties of the Dirac spinors and Dirac operators have fully been explored to date. Yet, it was less popular to view Eqs. (10) and (11) as a standard eigenvalue problem.[5] This is, however, not hard to imagine. It is because we immediately see that

$$\mathfrak{G} + \tilde{\mathfrak{G}} = -2mE, \tag{14}$$

where E is a (4, 4) identity matrix. We can at once obtain eigenvalues 0 and $-2m$ (each number doubly degenerate) with both $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$. On top of it, neither $\mathfrak{G}$ nor $\tilde{\mathfrak{G}}$ is a normal operator (i.e., neither Hermitian nor unitary). Hence, it is impossible to diagonalize $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ through the unitary similarity transformation.[6] Thus, these matrices are a little bit difficult to deal with in terms of matrix algebra. More specifically, it would be intractable to construct the diagonalizing matrix of the Dirac operators $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ by seeking the eigenvectors that belong to their eigenvalues.

Furthermore, spatial rotations and boosts are intricately mingled together so as to constitute the collective and general Lorentz transformation. In fact, to the best of the author's knowledge, to obtain direct solution of the Dirac equation by use of the matrix algebra has yet to be adequately investigated in a general form except for simple cases.[2]

In this article, the author wishes to show how the Dirac equation can be solved by directly transforming the matrices $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$. To this end, we determine the representation matrix $S(\Lambda)$ associated with the Lorentz group in the most general form and diagonalize the Dirac operators via the similarity transformation using $S(\Lambda)$. Their implications and significance are also discussed.

2 Transformation of space-time vector and Dirac spinor

Let us express a space-time vector X in the Minkowski space as

$$X = (e_0\ e_1\ e_2\ e_3)\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{15}$$

where $(e_0\ e_1\ e_2\ e_3)$ are the set of basis vectors of the Minkowski space. In terms of the special theory of relativity, a vector X is associated with a certain space-time point where a physical "event" has taken place. That event is observed and compared from the different inertial frames of reference that are connected to one another via the Lorentz transformations. To explicitly show this, we rewrite Eq. (15) as

$$X = (e_0\ e_1\ e_2\ e_3)\Lambda^{-1} \cdot \Lambda \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = (e'_0\ e'_1\ e'_2\ e'_3)\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}, \tag{16}$$

where Λ denotes a Lorentz transformation. Defining a shorthand notation in Eq. (16) such that

$$\boldsymbol{e} \equiv (e_0\ e_1\ e_2\ e_3),\ \boldsymbol{e}' \equiv (e'_0\ e'_1\ e'_2\ e'_3),\ x \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix},\ x' \equiv \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}, \tag{17}$$

we rewrite Eq. (16) as

$$X = \boldsymbol{e}x = \boldsymbol{e}\Lambda^{-1} \cdot \Lambda x = \boldsymbol{e}'x'. \tag{18}$$

The basis set $\boldsymbol{e}' = \boldsymbol{e}\Lambda^{-1}$ represents the set of basis vectors obtained by the Lorentz transformation Λ. The coordinates $x' = \Lambda x$ indicate those obtained by Λ as well.

Meanwhile, defining $\mathfrak{D} \equiv i\gamma^{\mu}\partial_{\mu} - m$, we rewrite Eq. (1) as

$$\mathfrak{D}\psi(x) = 0. \tag{19}$$

We assume that both the Dirac operator and Dirac spinor undergo some sort of transformation in connection with the Lorentz transformation.[1)] Operating $D(\Lambda)$ from the left on both sides of Eq. (19) and inserting $[D(\Lambda)]^{-1} \cdot D(\Lambda)$ between $\mathfrak{D}$ and $\psi(x)$, we obtain

$$D(\Lambda)\mathfrak{D}[D(\Lambda)]^{-1} \cdot D(\Lambda)\psi(x) = 0,$$

where $D(\Lambda)$ denotes a transformation operator associated with Λ. Further defining

$$\breve{\mathfrak{D}} \equiv D(\Lambda)\mathfrak{D}[D(\Lambda)]^{-1},\ \breve{\psi}(x') \equiv D(\Lambda)\psi(\Lambda^{-1}x'), \tag{20}$$

the Dirac equation is expressed in reference to the x'-system as

$$\breve{\mathfrak{D}}\breve{\psi}(x') = 0.$$

In Eq. (20) $\breve{\psi}(x')$ denotes the change in the functional form accompanied by the coordinate transformation.[1)] The whole collection of the Lorentz transformations forms the Lorentz group and, hence, we assume that $D(\Lambda)$ gives a representation pertinent to the Lorentz group. According to the custom,[1)] we define $S(\Lambda)$ as

$$S(\Lambda) \equiv D(\Lambda). \tag{21}$$

Regarding the plane waves, $\psi(x)$ is described by

$$\psi(x) = e^{\pm ipx}w(\boldsymbol{p}, h),$$

where $w(\boldsymbol{p}, h)$ represents either $u(\boldsymbol{p}, h)$ or $v(\boldsymbol{p}, h)$ of Eqs. (5) and (6). The quantity px ($\equiv p_{\mu}x^{\mu}$) in $e^{\pm ipx}$ is a scalar, and so invariant in relation to the Lorentz transformation with $p'x' = px$. Hence, it behaves as a constant in terms of the operation of $S(\Lambda)$.

In what follows, we examine how the Dirac equation is transformed by the Lorentz transformation to find the direct solution of the Dirac equation. We choose two coordinate systems for the inertial frames of reference. One is a frame where an electron is at rest (the x-system). In the other frame (x'-system), that electron is moving at a velocity $\boldsymbol{v}$. In the x-system, we have

$$\psi(x) = w(\mathbf{0}, h)e^{\pm ip_0x^0} = w(\mathbf{0}, h)e^{\pm imx^0}.$$

In the x'-system, in turn, the Dirac spinor is described by

$$\breve{\psi}(x') = e^{\pm imx^0} S(\Lambda) w(\mathbf{0}, h) = e^{\pm ip'x'} w(\boldsymbol{p}', h) = S(\Lambda)\psi(x). \tag{22}$$

3 Determination of general form of representation matrix $S(\Lambda)$

The plane-wave solutions of the Dirac equations [Eqs. (10) and (11)] are specified by momentum ($\boldsymbol{p}$) and helicity ($h = \pm 1$). To decide the direction of the normal to the wave front is our next task. It is easiest to solve the Dirac equation for an electron at rest. Let the inertial frame of reference to which the electron is at rest be O with the basis vectors given by $\boldsymbol{e}$. Let another inertial frame of reference where the electron is moving at a velocity $\boldsymbol{v}$ be O$'$ with the basis vectors of $\boldsymbol{e}'$ and the coordinate x' (see **Figure 1**). We assume that the propagation direction of the wave front parallels the direction of the electron motion. In Figure 1, the electron is moving in the direction specified by a zenithal angle θ $(0 \le \theta \le \pi)$ and azimuthal angle ϕ $(0 \le \phi \le 2\pi)$. Then, the transformation from the frame O to O$'$ is achieved via the following successive transformations:

(i) A Lorentz boost with $-\boldsymbol{v}$ along the x^3-axis,
(ii) a rotation around the x^2-axis by $-\theta$,
(iii) a rotation around the x'^3-axis by $-\phi$.

We adopt the "moving coordinate systems" with the different inertial frames of reference.[7] Note that the direction of the Lorentz boost ($-\boldsymbol{v}$) is opposite to that of the electron motion ($\boldsymbol{v}$). Thus, with the total transformation we obtain $\boldsymbol{e}' = \boldsymbol{e}\Lambda^{-1} = \boldsymbol{e}\Lambda_b \Lambda_\theta^{-1} \Lambda_\phi^{-1}$. Therefore, we have

$$\Lambda^{-1} = \Lambda_b \Lambda_\theta^{-1} \Lambda_\phi^{-1} \quad \text{i.e.,} \quad \Lambda = \Lambda_\phi \Lambda_\theta \Lambda_b^{-1}. \tag{23}$$

Finally, from Eq. (18) we get

$$x' = \Lambda x = \Lambda_\phi \Lambda_\theta \Lambda_b^{-1} x. \tag{24}$$

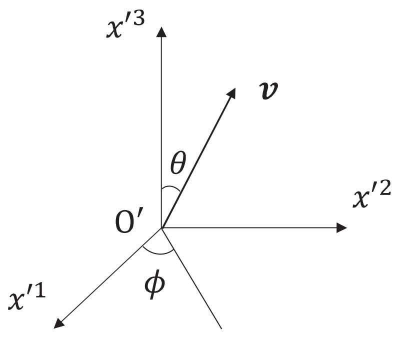

Figure 1 Geometry of the electron motion. The velocity of the electron is given by $\boldsymbol{v}$ and designated by a zenithal angle θ and azimuthal angle ϕ. The x'^1-, x'^2- and x'^3-axes are the spatial components of the frame O$'$.

Since $S(\Lambda)$ is the representation of the group, from Eqs. (21) and (23) we have

$$S(\Lambda) = S(\Lambda_\phi)S(\Lambda_\theta)S(\Lambda_b^{-1}) = S(\Lambda_\phi)S(\Lambda_\theta)[S(\Lambda_b)]^{-1}, \tag{25}$$

where Λ_ϕ, Λ_θ, and Λ_b are expressed as

$$\Lambda_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix},$$

$$\Lambda_b = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix}. \tag{26}$$

In Eq. (26), ω is said to be rapidity and defined as

$$\tanh\omega \equiv v \; (-\infty < \omega < \infty \Leftrightarrow -1 < v < 1), \tag{27}$$

where v is a velocity measured in a natural unit of the particle (i.e., electron).

As discussed in the previous section, to examine the constitution of the Dirac equation we wish to rewrite the Dirac equation and find the solutions with the electron at rest. Then, we construct the Dirac equation in the general case where the electron is moving as shown in Figure 1.

The properties of $S(\Lambda)$ have been fully investigated, and so we borrow their matrix representations from literature.[8)] We have

$$S(\Lambda_\phi) = \exp(\phi\rho_3) = \begin{pmatrix} e^{i\phi/2} & 0 & 0 & 0 \\ 0 & e^{-i\phi/2} & 0 & 0 \\ 0 & 0 & e^{i\phi/2} & 0 \\ 0 & 0 & 0 & e^{-i\phi/2} \end{pmatrix}, \tag{28}$$

$$S(\Lambda_\theta) = \exp(\theta\rho_2) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} & 0 & 0 \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ 0 & 0 & -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{29}$$

with ρ_m given by

$$\rho_m \equiv (-i)J^{kl};\ J^{kl} \equiv \frac{i}{4}[\gamma^k, \gamma^l]\ (k, l, m = 1, 2, 3). \tag{30}$$

In Eq. (30) $\gamma^k\ (k = 1, 2, 3)$ are the gamma matrices and $k, l,$ and m change cyclic. We have

$$J^{kl} = \frac{1}{2}\begin{pmatrix} \sigma_m & \mathbf{0} \\ \mathbf{0} & \sigma_m \end{pmatrix},$$

where σ_m is a Pauli spin matrix and $\mathbf{0}$ denotes a (2, 2) zero matrix. Moreover, we have

$$[S(\Lambda_b)]^{-1} = \exp(\omega\beta_3) = \begin{pmatrix} \cosh\frac{\omega}{2} & 0 & -\sinh\frac{\omega}{2} & 0 \\ 0 & \cosh\frac{\omega}{2} & 0 & \sinh\frac{\omega}{2} \\ -\sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} & 0 \\ 0 & \sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} \end{pmatrix} \tag{31}$$

with β_3 given by $\beta_3 \equiv (-i)\,J^{03};\ J^{03} \equiv \frac{i}{2}\gamma^0\gamma^3$. Consequently, we get

$$S(\Lambda) = S(\Lambda_\phi)S(\Lambda_\theta)[S(\Lambda_b)]^{-1} =$$

$$\begin{pmatrix} e^{i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} & -e^{i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} & e^{i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} \\ -e^{-i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{-i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{-i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} & e^{-i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} \\ -e^{i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} & e^{i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} & e^{i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} \\ e^{-i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} & e^{-i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{-i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} \end{pmatrix}. \tag{32}$$

In the next section, we make the most of Eq. (32) to directly solve the Dirac equations Eqs. (10) and (11) to get their plane-wave solutions.

4 Solutions of the Dirac equation

In the frame O where the electron is at rest, Eqs. (10) and (11) take a particularly simple but important form. Since we have $\boldsymbol{p} = \mathbf{0}$, from Eq. (7) we obtain $p^0 = m\ (> 0)$, and so Eqs. (10) and (11) are respectively reduced to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2m & 0 \\ 0 & 0 & 0 & -2m \end{pmatrix} u(\mathbf{0},h) = 0 \quad \text{with} \quad A \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2m & 0 \\ 0 & 0 & 0 & -2m \end{pmatrix} \tag{33}$$

and

$$\begin{pmatrix} -2m & 0 & 0 & 0 \\ 0 & -2m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v(\mathbf{0},h) = 0 \quad \text{with} \quad B \equiv \begin{pmatrix} -2m & 0 & 0 & 0 \\ 0 & -2m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

The matrices A of Eq. (33) and B of Eq. (34) correspond to the positive-energy or negative-energy solution, respectively. Note that both A and B have been diagonalized. Consequently, if the Dirac operator $\mathfrak{G}$ of Eq. (12) and $\tilde{\mathfrak{G}}$ of Eq. (13) are related to A and B, respectively, this should lead to the desired solutions of the Dirac equation.

Equations (33) and (34) are immediately solved to give[1)]

$$u(\mathbf{0},-1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u(\mathbf{0},+1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \tag{35}$$

$$v(\mathbf{0},+1) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v(\mathbf{0},-1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{36}$$

The minus sign of RHS for the second equation of Eq. (36) is due to the charge conjugation.[1)] Operating $S(\Lambda)$ given in Eq. (32) from the left on both sides of Eq. (33) and inserting $[S(\Lambda)]^{-1} \cdot S(\Lambda)$ between A and $u(\mathbf{0},h)$, we obtain

$$S(\Lambda)A[S(\Lambda)]^{-1} \cdot S(\Lambda)u(\mathbf{0},\pm 1) = 0. \tag{37}$$

Then, we should be able to get the Dirac operator $S(\Lambda)A[S(\Lambda)]^{-1}$ and the corresponding Dirac spinor solution $S(\Lambda)u(\mathbf{0},\pm 1)$. We have

$$[S(\Lambda)]^{-1} = S(\Lambda_b)[S(\Lambda_\theta)]^{-1}[S(\Lambda_\phi)]^{-1} =$$

$$\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} & -e^{i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{-i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} & -e^{i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} \\ e^{-i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} & -e^{i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} \\ e^{-i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} & -e^{i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} & e^{-i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} & -e^{i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} \\ -e^{-i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} & -e^{i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} & e^{-i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} & e^{i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} \end{pmatrix}. \tag{38}$$

Hence, we obtain

$$S(\Lambda)A[S(\Lambda)]^{-1} = (-2m) \times$$

$$\begin{pmatrix} -\sinh^2\frac{\omega}{2} & 0 & -\cos\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} & e^{i\phi}\sin\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} \\ 0 & -\sinh^2\frac{\omega}{2} & e^{-i\phi}\sin\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} & \cos\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} \\ \cos\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} & -e^{i\phi}\sin\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} & \cosh^2\frac{\omega}{2} & 0 \\ -e^{-i\phi}\sin\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} & -\cos\theta\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} & 0 & \cosh^2\frac{\omega}{2} \end{pmatrix}. \tag{39}$$

Now, using the formulae of the hyperbolic functions

$$\cosh\omega = \frac{1}{\sqrt{1-\tanh^2\omega}}, \tag{40}$$

we have

$$\cosh\omega = 1/\sqrt{1-v^2} \equiv \gamma. \tag{41}$$

Meanwhile, we have[9)]

$$p'^0 = m\gamma = m\cosh\omega,\ \boldsymbol{p}' = m\gamma\boldsymbol{v}, \tag{42}$$

the first equation of which represents the equivalence theorem of mass and energy due to Einstein. Also, using formulae of the hyperbolic functions we have

$$\cosh\frac{\omega}{2} = \sqrt{\frac{1+\cosh\omega}{2}} = \sqrt{\frac{p'^0+m}{2m}},\ \sinh\frac{\omega}{2} = \sqrt{\frac{-1+\cosh\omega}{2}} = \sqrt{\frac{p'^0-m}{2m}},$$

$$\tanh\frac{\omega}{2} = \frac{\sinh\frac{\omega}{2}}{\cosh\frac{\omega}{2}} = \sqrt{(p'^0-m)/(p'^0+m)} = |\boldsymbol{p}'|/(p'^0+m). \tag{43}$$

From Eqs. (7) and (43), we obtain

$$\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} = \frac{1}{2m}\sqrt{(p'^0+m)(p'^0-m)} = \frac{1}{2m}|\boldsymbol{p}'|. \tag{44}$$

Using the relations Eqs. (40)–(44), finally we get

$$S(\Lambda)A[S(\Lambda)]^{-1} = \begin{pmatrix} p'^0 - m & 0 & |\boldsymbol{p}'|\cos\theta & -|\boldsymbol{p}'|e^{i\phi}\sin\theta \\ 0 & p'^0 - m & -|\boldsymbol{p}'|e^{-i\phi}\sin\theta & -|\boldsymbol{p}'|\cos\theta \\ -|\boldsymbol{p}'|\cos\theta & |\boldsymbol{p}'|e^{i\phi}\sin\theta & -p'^0 - m & 0 \\ |\boldsymbol{p}'|e^{-i\phi}\sin\theta & |\boldsymbol{p}'|\cos\theta & 0 & -p'^0 - m \end{pmatrix}. \tag{45}$$

Converting the polar coordinate into the Cartesian coordinate, we find that Eq. (45) is identical with $\mathfrak{G}$ defined in Eq. (12). That is, we have

$$S(\Lambda)A[S(\Lambda)]^{-1} = \mathfrak{G}. \tag{46}$$

From Eq. (37), in turn, we obtain

$$S(\Lambda)u(\boldsymbol{0},-1) = \begin{pmatrix} e^{i\phi/2}\cos\frac{\theta}{2}\cosh\frac{\omega}{2} \\ -e^{-i\phi/2}\sin\frac{\theta}{2}\cosh\frac{\omega}{2} \\ -e^{i\phi/2}\cos\frac{\theta}{2}\sinh\frac{\omega}{2} \\ e^{-i\phi/2}\sin\frac{\theta}{2}\sinh\frac{\omega}{2} \end{pmatrix} = \sqrt{\frac{p'^0+m}{2m}}\begin{pmatrix} e^{i\phi/2}\cos\frac{\theta}{2} \\ -e^{-i\phi/2}\sin\frac{\theta}{2} \\ -\mathcal{S}e^{i\phi/2}\cos\frac{\theta}{2} \\ \mathcal{S}e^{-i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}, \tag{47}$$

where we define $\mathcal{S}$ as

$$\mathcal{S} \equiv |\boldsymbol{p}'|/(p'^0+m) = \tanh\frac{\omega}{2}. \tag{48}$$

Thus, as the full description of the plane-wave solution we get e.g.,

$$\breve{\psi}_1(x') \equiv e^{-ip'x'} S(\Lambda) u(\mathbf{0}, -1) = e^{-ip'x'} \sqrt{\frac{p'^0+m}{2m}} \begin{pmatrix} e^{i\phi/2} \cos\frac{\theta}{2} \\ -e^{-i\phi/2} \sin\frac{\theta}{2} \\ -\mathcal{S} e^{i\phi/2} \cos\frac{\theta}{2} \\ \mathcal{S} e^{-i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}. \quad (49)$$

Similarly, we have

$$\breve{\psi}_2(x') \equiv e^{-ip'x'} S(\Lambda) u(\mathbf{0}, +1) = e^{-ip'x'} \begin{pmatrix} e^{i\phi/2} \sin\frac{\theta}{2} \cosh\frac{\omega}{2} \\ e^{-i\phi/2} \cos\frac{\theta}{2} \cosh\frac{\omega}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \sinh\frac{\omega}{2} \\ e^{-i\phi/2} \cos\frac{\theta}{2} \sinh\frac{\omega}{2} \end{pmatrix} = e^{-ip'x'} \sqrt{\frac{p'^0+m}{2m}} \begin{pmatrix} e^{i\phi/2} \sin\frac{\theta}{2} \\ e^{-i\phi/2} \cos\frac{\theta}{2} \\ \mathcal{S} e^{i\phi/2} \sin\frac{\theta}{2} \\ \mathcal{S} e^{-i\phi/2} \cos\frac{\theta}{2} \end{pmatrix}. \quad (50)$$

Also, from Eqs. (14), (33), and (34), we readily confirm that

$$S(\Lambda) B [S(\Lambda)]^{-1} = \widetilde{\mathfrak{G}}. \quad (51)$$

Similarly to Eq. (37), we have

$$S(\Lambda) B [S(\Lambda)]^{-1} \cdot S(\Lambda) v(\mathbf{0}, \mp 1) = 0. \quad (52)$$

Consequently, we obtain

$$\breve{\psi}_3(x') \equiv e^{ip'x'} S(\Lambda) v(\mathbf{0}, +1) = e^{ip'x'} \begin{pmatrix} -e^{i\phi/2} \cos\frac{\theta}{2} \sinh\frac{\omega}{2} \\ e^{-i\phi/2} \sin\frac{\theta}{2} \sinh\frac{\omega}{2} \\ e^{i\phi/2} \cos\frac{\theta}{2} \cosh\frac{\omega}{2} \\ -e^{-i\phi/2} \sin\frac{\theta}{2} \cosh\frac{\omega}{2} \end{pmatrix} = e^{ip'x'} \sqrt{\frac{p'^0+m}{2m}} \begin{pmatrix} -\mathcal{S} e^{i\phi/2} \cos\frac{\theta}{2} \\ \mathcal{S} e^{-i\phi/2} \sin\frac{\theta}{2} \\ e^{i\phi/2} \cos\frac{\theta}{2} \\ -e^{-i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}, \quad (53)$$

$$\breve{\psi}_4(x') \equiv e^{ip'x'} S(\Lambda) v(\mathbf{0}, -1) = e^{ip'x'} \begin{pmatrix} -e^{i\phi/2} \sin\frac{\theta}{2} \sinh\frac{\omega}{2} \\ -e^{-i\phi/2} \cos\frac{\theta}{2} \sinh\frac{\omega}{2} \\ -e^{i\phi/2} \sin\frac{\theta}{2} \cosh\frac{\omega}{2} \\ -e^{-i\phi/2} \cos\frac{\theta}{2} \cosh\frac{\omega}{2} \end{pmatrix} = e^{ip'x'} \sqrt{\frac{p'^0+m}{2m}} \begin{pmatrix} -\mathcal{S} e^{i\phi/2} \sin\frac{\theta}{2} \\ -\mathcal{S} e^{-i\phi/2} \cos\frac{\theta}{2} \\ -e^{i\phi/2} \sin\frac{\theta}{2} \\ -e^{-i\phi/2} \cos\frac{\theta}{2} \end{pmatrix}. \quad (54)$$

The above results shown in Eqs. (49), (50), (53), and (54) are consistent with those described in the literature.[10] This implies that Eqs. (37) and (52) properly express the Dirac equation in the representation space related to the frame O'. On the basis of the discussion developed in this section, we conclude that Eqs. (37) and (52) are equivalent to Eqs. (10) and (11), respectively. Hence, we obtain

$$S(\Lambda)u(\mathbf{0},h) = u(\boldsymbol{p}',h),\ S(\Lambda)v(\mathbf{0},h) = v(\boldsymbol{p}',h). \tag{55}$$

Multiplying the exponential term e^{-imx^0} on both sides of Eq. (37) or e^{imx^0} on both sides of Eq. (52) and using the notation of Eq. (22), we get a succinct form expressed as

$$\breve{\mathfrak{G}}\breve{\psi}(x') = 0, \tag{56}$$

where $\breve{\mathfrak{G}}$ represents either $\mathfrak{G}$ of Eq. (46) or $\tilde{\mathfrak{G}}$ of Eq. (51); $\breve{\psi}(x')$ is chosen from among $\breve{\psi}_1(x')$, $\breve{\psi}_2(x')$, $\breve{\psi}_3(x')$, and $\breve{\psi}_4(x')$ obtained above.

Rewriting Eq. (46), we have

$$[S(\Lambda)]^{-1}\mathfrak{G}S(\Lambda) = A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2m & 0 \\ 0 & 0 & 0 & -2m \end{pmatrix}. \tag{57}$$

Also, rewriting Eq. (51) we have

$$[S(\Lambda)]^{-1}\tilde{\mathfrak{G}}S(\Lambda) = B = \begin{pmatrix} -2m & 0 & 0 & 0 \\ 0 & -2m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{58}$$

Equations (57) and (58) clearly indicate that $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ have been diagonalized through the similarity transformation using $S(\Lambda)$ to produce A and B, respectively. Namely, the matrices $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ are semi-simple (or diagonalizable).[11] At the same time, the diagonalizing matrix $S(\Lambda)$ yields the desired solutions of the Dirac equation as can be seen in Eq. (55).

In summary, once we can find the proper representation matrix $S(\Lambda)$ by calculating Eq. (32), the Dirac operators can automatically be diagonalized and we are able to determine the "eigenspinors" of the Dirac equation.

In terms of the group theory, Eq. (46) indicates that the Dirac operators $\mathfrak{G}$ and A are conjugate to each other.[12] In turn, Eq. (51) implies that $\tilde{\mathfrak{G}}$ and B are conjugate to each other. Correspondingly, Eq. (55) represents the transformation between the Dirac spinors. Thus, the Dirac equation of Eq. (56) that describes the plane wave in the frame O' is connected to Eq. (33) or (34) that is pertinent to the plane wave in O (i.e., the rest frame) through the medium of the matrix $S(\Lambda)$.

In the discussions made thus far, we have dealt with the Dirac equation in the relationship between the rest frame of the electron (the frame O) and the moving frame O'. Next, we wish to generalize the situation by considering the relationship between two moving frames. Suppose that we have another inertial frame of reference $\widehat{\mathrm{O}}$ where the electron is moving at a velocity $\hat{\boldsymbol{v}}$ with a zenithal angle $\hat{\theta}$ and azimuthal angle $\hat{\phi}$ (see Figure 1).

To address the issue appropriately, we break down the problem again into the relationship between the rest frame and the moving frame of the electron. In other words, considering the relationship between O and $\widehat{\mathrm{O}}$ as well as that between O and O', we relate the moving frame $\widehat{\mathrm{O}}$ to the other moving frame O' through the medium of the rest frame O. Let

$$\hat{x} \equiv \begin{pmatrix} \widehat{x^0} \\ \widehat{x^1} \\ \widehat{x^2} \\ \widehat{x^3} \end{pmatrix}$$

be the coordinates of the frame $\widehat{\mathrm{O}}$. Defining $\widehat{\Lambda}$ as below and following Eq. (24), $\hat{x}$ is given by

$$\hat{x} = \widehat{\Lambda} x = \Lambda_{\hat{\phi}} \Lambda_{\hat{\theta}} \Lambda_{\hat{b}}^{-1} x,$$

where $\Lambda_{\hat{\phi}}$, $\Lambda_{\hat{\theta}}$, and $\Lambda_{\hat{b}}$ can be obtained by replacing ϕ, θ, and ω of Eq. (26) with $\hat{\phi}$, $\hat{\theta}$, and $\hat{\omega}$, respectively. The rapidity $\hat{\omega}$ is defined as in Eq. (27) from the velocity $\hat{\boldsymbol{v}}$ of the electron measured in the frame $\widehat{\mathrm{O}}$ (also, see Figure 1). Using Eq. (24), we get

$$\hat{x} = \widehat{\Lambda} \Lambda^{-1} x'. \tag{59}$$

Thus, the Lorentz transformation that links the frame $\widehat{\mathrm{O}}$ to O' is described by $\widehat{\Lambda}\Lambda^{-1}$ through the mediation of the rest frame O. Meanwhile, using the notation of Eq. (22) we have

$$\hat{\psi}(\hat{x}) = S(\widehat{\Lambda})\psi(x) = e^{\pm imx^0} S(\widehat{\Lambda}) w(\mathbf{0}, h) = e^{\pm imx^0} S(\widehat{\Lambda}) [S(\Lambda)]^{-1} S(\Lambda) w(\mathbf{0}, h)$$

$$= e^{\pm imx^0} S(\widehat{\Lambda}) S(\Lambda^{-1}) S(\Lambda) w(\mathbf{0}, h) = e^{\pm imx^0} S(\widehat{\Lambda}\Lambda^{-1}) S(\Lambda) w(\mathbf{0}, h)$$

$$= S(\widehat{\Lambda}\Lambda^{-1}) \breve{\psi}(x'), \tag{60}$$

where with the fourth and fifth equalities we used the fact that $S(\Lambda)$ is the representation of the group; the last equality resulted from Eq. (22). The function $\hat{\psi}(\hat{x})$ is the Dirac spinor defined in the moving frame $\widehat{\mathrm{O}}$. Since $\widehat{\Lambda}$ and Λ are the elements of the Lorentz group, so is $\widehat{\Lambda}\Lambda^{-1}$.

In Eq. (60), $S(\hat{\Lambda})$ is obtained by replacing ϕ, θ, and ω of Eq. (32) with $\hat{\phi}$, $\hat{\theta}$, and $\hat{\omega}$, respectively. Thus, the Dirac spinor is transformed according to $S(\hat{\Lambda}\Lambda^{-1})$ in correspondence to the Lorentz transformation $\hat{\Lambda}\Lambda^{-1}$ described in Eq. (59). More importantly, Eq. (60) represents the transformation of the Dirac spinor between two arbitrarily chosen inertial frames of reference. Note, at the same time, that Eqs. (59) and (60) can be rewritten in a complementary way as $x' = \Lambda(\hat{\Lambda})^{-1}\hat{x}$ and $\breve{\psi}(x') = S[\Lambda(\hat{\Lambda})^{-1}]\hat{\psi}(\hat{x})$, respectively.

Explicit matrix representations of $\hat{\Lambda}\Lambda^{-1}$ of Eq. (59) and $S(\hat{\Lambda}\Lambda^{-1})$ of Eq. (60) are listed at the end of the present article (see **Appendix**). In Appendix, if we put $\phi = \theta = \omega = 0$, $S(\hat{\Lambda}\Lambda^{-1})$ is identical with a matrix obtained by replacing ϕ, θ, and ω with $\hat{\phi}$, $\hat{\theta}$, and $\hat{\omega}$, respectively, in RHS of Eq. (32). If we put $\hat{\phi} = \hat{\theta} = \hat{\omega} = 0$, in turn, $S(\hat{\Lambda}\Lambda^{-1})$ is identical to RHS of Eq. (38).

In accordance with Eqs. (46) and (56), the Dirac equation in the frame $\hat{\mathrm{O}}$ is expressed as

$$\hat{\mathfrak{G}}\hat{\psi}(\hat{x}) = 0, \tag{61}$$

where $\hat{\mathfrak{G}}$ is given by either $\hat{\mathfrak{G}} = S(\hat{\Lambda})A[S(\hat{\Lambda})]^{-1}$ or $\hat{\mathfrak{G}} = S(\hat{\Lambda})B[S(\hat{\Lambda})]^{-1}$. Using Eqs. (57) and (58) along with Eq. (60), Eq. (61) can be rewritten as

$$S(\hat{\Lambda})[S(\Lambda)]^{-1}\breve{\mathfrak{G}}S(\Lambda)[S(\hat{\Lambda})]^{-1} \cdot S(\hat{\Lambda})[S(\Lambda)]^{-1}\breve{\psi}(x') = S(\hat{\Lambda})[S(\Lambda)]^{-1}\breve{\mathfrak{G}}\breve{\psi}(x') = 0, \tag{62}$$

where with the LHS once again we used the fact that $S(\Lambda)$ is the representation of the group. Thus, we find that Eq. (62) is equivalent to Eq. (56).

Equations (56) and (61) clearly indicate that the Dirac equation is transformed between two arbitrarily chosen inertial frames of reference via $S(\Xi)$ in which Ξ denotes a Lorentz transformation that links the said two frames. As a special case, one out of the two inertial frames of reference can be the frame where the electron stays at rest. In that case, $S(\Xi)$ is a diagonalizing matrix of the Dirac operator. Namely, the Dirac operator can be diagonalized through the similarity transformation based on $S(\Xi)$.

Meanwhile, we have

$$\hat{\mathfrak{G}} = S(\hat{\Lambda}\Lambda^{-1})\breve{\mathfrak{G}}[S(\hat{\Lambda}\Lambda^{-1})]^{-1}. \tag{63}$$

This gives the transformation of the Dirac operator between the moving frames $\hat{\mathrm{O}}$ and O'. If Λ is the identity operator, the frame O' is identical to the rest frame O and Eq. (63) is reduced to

$$\hat{\mathfrak{G}} = S(\hat{\Lambda})\breve{\mathfrak{G}}[S(\hat{\Lambda})]^{-1} \tag{64}$$

with $\breve{\mathfrak{G}} = A$ or $\breve{\mathfrak{G}} = B$. Then, Eq. (64) is virtually the same as Eq. (46) or Eq. (51). If $\hat{\Lambda}$ is the identity operator, in turn, the frame $\hat{\mathrm{O}}$ is identical to the frame O, and so Eq. (63) can be rewritten as $\hat{\mathfrak{G}} = [S(\Lambda)]^{-1}\breve{\mathfrak{G}}S(\Lambda)$ with $\hat{\mathfrak{G}} = A$ or $\hat{\mathfrak{G}} = B$ so as to be reduced to Eq. (57) or Eq. (58).

5 Polar decomposition of the Dirac operators and related matrix algebra

So far, we have examined the transformation properties of the Dirac equation in terms of the Dirac spinors and Dirac operators. These spinors and operators provide a unique opportunity to study further important aspects from the point of view of matrix algebra.

One of interesting topics lies in the polar decomposition of a matrix. For this, we have a following theorem.[6)]

Theorem 1[6)]

Let A be a non-singular matrix. Then, there exist positive definite Hermitian matrices H_1 and H_2 as well as a unitary matrix U such that

$$A = UH_1 = H_2U. \tag{65}$$

If and only if A is a normal matrix, then we have $H_1 = H_2$. (That is, $H_1 = H_2$ and U are commutative.)

Equation (65) is said to be a polar decomposition and Theorem 1 implies that the decomposition of Eq. (65) is unique. Regarding the polar decomposition, we have a good example with the representation matrix $S(\Lambda)$. Using Eqs. (28), (29), and (31), we describe $S(\Lambda)$ of Eq. (32) in such a way that

$$S(\Lambda) =$$

$$\begin{pmatrix} e^{i\phi/2}\cos\frac{\theta}{2} & e^{i\phi/2}\sin\frac{\theta}{2} & 0 & 0 \\ -e^{-i\phi/2}\sin\frac{\theta}{2} & e^{-i\phi/2}\cos\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & e^{i\phi/2}\cos\frac{\theta}{2} & e^{i\phi/2}\sin\frac{\theta}{2} \\ 0 & 0 & -e^{-i\phi/2}\sin\frac{\theta}{2} & e^{-i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cosh\frac{\omega}{2} & 0 & -\sinh\frac{\omega}{2} & 0 \\ 0 & \cosh\frac{\omega}{2} & 0 & \sinh\frac{\omega}{2} \\ -\sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} & 0 \\ 0 & \sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} \end{pmatrix}. \tag{66}$$

Putting $S(\Lambda_u) \equiv S(\Lambda_\phi)S(\Lambda_\theta) = S(\Lambda_\phi\Lambda_\theta)$, with the polar decomposition of Eq. (66) we obtain

$$S(\Lambda) = S(\Lambda_u)[S(\Lambda_b)]^{-1}. \tag{67}$$

In RHS of Eq. (66), the first matrix $S(\Lambda_u)$ is unitary and the second matrix $[S(\Lambda_b)]^{-1}$ is Hermitian. The eigenvalues of $[S(\Lambda_b)]^{-1}$ are

$$\cosh\frac{\omega}{2}+\sinh\frac{\omega}{2}\ \text{(doubly degenerate)},\ \cosh\frac{\omega}{2}-\sinh\frac{\omega}{2}\ \text{(doubly degenerate)}$$

with a determinant of $1\ (>0)$. Hence, the Hermitian matrix $[S(\Lambda_b)]^{-1}$ is indeed positive definite.

Suppose that another decomposition is given by

$$S(\Lambda) = \widetilde{H} S(\Lambda_u). \tag{68}$$

Then, we have

$$\widetilde{H} = S(\Lambda_u)[S(\Lambda_b)]^{-1}[S(\Lambda_u)]^{\dagger}. \tag{69}$$

Since the unitary similarity transformation of an Hermitian matrix retains the Hermiticity and holds eigenvalues of that matrix unchanged, from Eq. (69) $\widetilde{H}$ is positive definite Hermitian as well. Hence, from the uniqueness of the polar decomposition we find that Eq. (68) is certainly another polar decomposition. The matrix $\widetilde{H}$ is given by

$$\widetilde{H} = \begin{pmatrix} \cosh\frac{\omega}{2} & 0 & -\cos\theta\sinh\frac{\omega}{2} & e^{i\phi}\sin\theta\sinh\frac{\omega}{2} \\ 0 & \cosh\frac{\omega}{2} & e^{-i\phi}\sin\theta\sinh\frac{\omega}{2} & \cos\theta\sinh\frac{\omega}{2} \\ -\cos\theta\sinh\frac{\omega}{2} & e^{i\phi}\sin\theta\sinh\frac{\omega}{2} & \cosh\frac{\omega}{2} & 0 \\ e^{-i\phi}\sin\theta\sinh\frac{\omega}{2} & \cos\theta\sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} \end{pmatrix}. \tag{70}$$

Notice that we get the representation $\widetilde{H}$ as a result of viewing the operator $[S(\Lambda_b)]^{-1}$ in reference to the frame O'. The trace of $\widetilde{H}$ is $4\cosh\frac{\omega}{2}$, which is held unchanged after the unitary similarity transformation with $S(\Lambda_u)$. Let us think of the following simple example.

Example 1

In the general case of the representation matrix described by Eq. (32) of the previous section, consider a special case of $\theta = 0$. In that case, from Eqs. (32) and (66) we obtain

$$\begin{pmatrix} e^{i\phi/2}\cosh\frac{\omega}{2} & 0 & -e^{i\phi/2}\sinh\frac{\omega}{2} & 0 \\ 0 & e^{-i\phi/2}\cosh\frac{\omega}{2} & 0 & e^{-i\phi/2}\sinh\frac{\omega}{2} \\ -e^{i\phi/2}\sinh\frac{\omega}{2} & 0 & e^{i\phi/2}\cosh\frac{\omega}{2} & 0 \\ 0 & e^{-i\phi/2}\sinh\frac{\omega}{2} & 0 & e^{-i\phi/2}\cosh\frac{\omega}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\phi/2} & 0 & 0 & 0 \\ 0 & e^{-i\phi/2} & 0 & 0 \\ 0 & 0 & e^{i\phi/2} & 0 \\ 0 & 0 & 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cosh\frac{\omega}{2} & 0 & -\sinh\frac{\omega}{2} & 0 \\ 0 & \cosh\frac{\omega}{2} & 0 & \sinh\frac{\omega}{2} \\ -\sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} & 0 \\ 0 & \sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} \end{pmatrix}. \quad (71)$$

The LHS of Eq. (71) is a normal matrix and the two matrices of RHS are commutative as expected. As in this example, if an axis of the rotation and a direction of the Lorentz boost coincide, such successive operations of the rotation and boost are commutative and, hence, the relevant representation matrix $S(\Lambda)$ is normal according to Theorem 1.

Because of the presence of the Lorentz boost, $S(\Lambda)$ is in general not unitary. As is evident from Eq. (27), the Lorentz group is non-compact. In such a case, the representation matrix cannot be made unitary. In this respect, the representation matrix $S(\Lambda)$ is a typical illustration.

Another interesting aspect of the Dirac operators lies in the fact that those operators act as projection operators. In fact, from Eq. (14) we have

$$\frac{\mathfrak{G}}{(-2m)} + \frac{\tilde{\mathfrak{G}}}{(-2m)} = E. \quad (72)$$

Defining $\mathfrak{P}$ and $\mathfrak{Q}$ as

$$\mathfrak{P} \equiv \frac{\tilde{\mathfrak{G}}}{(-2m)}, \ \mathfrak{Q} \equiv \frac{\mathfrak{G}}{(-2m)}, \quad (73)$$

we have

$$\mathfrak{P} + \mathfrak{Q} = E. \quad (74)$$

Also, we obtain

$$\mathfrak{P}^2 = \frac{SBS^{-1}}{(-2m)} \cdot \frac{SBS^{-1}}{(-2m)} = \frac{SB^2S^{-1}}{(-2m)^2} = \frac{(-2m)SBS^{-1}}{(-2m)^2} = \frac{SBS^{-1}}{(-2m)} = \frac{\tilde{\mathfrak{G}}}{(-2m)} = \mathfrak{P},$$

$$\mathfrak{Q}^2 = \frac{SAS^{-1}}{(-2m)} \cdot \frac{SAS^{-1}}{(-2m)} = \frac{SA^2S^{-1}}{(-2m)^2} = \frac{(-2m)SAS^{-1}}{(-2m)^2} = \frac{SAS^{-1}}{(-2m)} = \frac{\mathfrak{G}}{(-2m)} = \mathfrak{Q},$$

$$\mathfrak{P}\mathfrak{Q} = \frac{SBS^{-1}}{(-2m)} \cdot \frac{SAS^{-1}}{(-2m)} = \frac{SBAS^{-1}}{(-2m)^2} = 0. \quad (75)$$

The properties represented by Eqs. (74) and (75) reflect the nature of $\mathfrak{P}$ and $\mathfrak{Q}$ as projection operators.[13] In connection with the *Hermitian* projection operators, various aspects have fully been explored in the literature.[13] Nonetheless, as neither $\mathfrak{P}$ nor $\mathfrak{Q}$ is an Hermitian operator, special care should be taken.

6 Conclusion

The author has developed a method to find the direct solution of the Dirac equation. The essential point rests upon the fact that we have determined the representation matrix $S(\Lambda)$ in the most general form in the representation space of the Dirac spinors. The present method helps study the constitution and the transformation properties of the Dirac spinors and Dirac operators.

Furthermore, we can make the most of the method to address the topics of matrix algebra such as the polar decomposition. The characteristics of the Dirac operators as the projection operators are of great importance and interest as well.

7 References

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Appendix:

$\hat{\Lambda}\Lambda^{-1}$ of Eq. (59) =

$$\begin{pmatrix} \cosh(\hat{\omega}-\omega) & \cos\phi\sin\theta\sinh(\hat{\omega}-\omega) & \sin\phi\sin\theta\sinh(\hat{\omega}-\omega) & \cos\theta\sinh(\hat{\omega}-\omega) \\ \cos\hat{\phi}\sin\hat{\theta}\sinh(\hat{\omega}-\omega) & \begin{matrix}\cos\hat{\phi}\sin\hat{\theta}\cos\phi\sin\theta\cosh(\hat{\omega}-\omega) \\ +\cos\hat{\phi}\cos\hat{\theta}\cos\phi\cos\theta+\sin\hat{\phi}\sin\phi\end{matrix} & \begin{matrix}\cos\hat{\phi}\sin\hat{\theta}\sin\phi\sin\theta\cosh(\hat{\omega}-\omega) \\ +\cos\hat{\phi}\cos\hat{\theta}\sin\phi\cos\theta-\sin\hat{\phi}\cos\phi\end{matrix} & \begin{matrix}\cos\hat{\phi}\sin\hat{\theta}\cos\theta\cosh(\hat{\omega}-\omega) \\ -\cos\hat{\phi}\cos\hat{\theta}\sin\theta\end{matrix} \\ \sin\hat{\phi}\sin\hat{\theta}\sinh(\hat{\omega}-\omega) & \begin{matrix}\sin\hat{\phi}\sin\hat{\theta}\cos\phi\sin\theta\cosh(\hat{\omega}-\omega) \\ +\sin\hat{\phi}\cos\hat{\theta}\cos\phi\cos\theta-\cos\hat{\phi}\sin\phi\end{matrix} & \begin{matrix}\sin\hat{\phi}\sin\hat{\theta}\sin\phi\sin\theta\cosh(\hat{\omega}-\omega) \\ +\sin\hat{\phi}\cos\hat{\theta}\sin\phi\cos\theta+\cos\hat{\phi}\cos\phi\end{matrix} & \begin{matrix}\sin\hat{\phi}\sin\hat{\theta}\cos\theta\cosh(\hat{\omega}-\omega) \\ -\sin\hat{\phi}\cos\hat{\theta}\sin\theta\end{matrix} \\ \cos\hat{\theta}\sinh(\hat{\omega}-\omega) & \begin{matrix}\cos\hat{\theta}\cos\phi\sin\theta\cosh(\hat{\omega}-\omega) \\ -\sin\hat{\theta}\cos\phi\cos\theta\end{matrix} & \begin{matrix}\cos\hat{\theta}\sin\phi\sin\theta\cosh(\hat{\omega}-\omega) \\ -\sin\hat{\theta}\sin\phi\cos\theta\end{matrix} & \begin{matrix}\cos\hat{\theta}\cos\theta\cosh(\hat{\omega}-\omega) \\ +\sin\hat{\theta}\sin\theta\end{matrix} \end{pmatrix}$$

$S\left(\hat{\Lambda}\Lambda^{-1}\right)$ of Eq. (60) =

$$\begin{pmatrix} e^{i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} & e^{i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} & -e^{i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} & e^{i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} \\ -e^{-i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} & e^{-i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} & e^{-i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} & e^{-i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} \\ -e^{i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} & e^{i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} & e^{i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} & e^{i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} \\ e^{-i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} & e^{-i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}+\theta}{2}\sinh\frac{\hat{\omega}-\omega}{2} & -e^{-i(\hat{\phi}+\phi)/2}\sin\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} & e^{-i(\hat{\phi}-\phi)/2}\cos\frac{\hat{\theta}-\theta}{2}\cosh\frac{\hat{\omega}-\omega}{2} \end{pmatrix}$$