A condition for a circle domain and an infinitely generated classical Schottky group

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1 Introduction

Let Ω^* be a domain in $\widehat{\mathbb{C}}$ that contains ∞ and $\{\Omega_n\}$ be a regular exhaustion of Ω^*, i.e.,

(1) the boundary $\partial\Omega_n$ of Ω_n consists of a finite number of analytic Jordan curves,

(2) every component of $\Omega^* - \Omega_n$ is non compact,

(3) $(\Omega_n \cup \partial\Omega_n) \subset \Omega_{n+1}$,

(4) $\bigcup_{n=1}^{\infty} \Omega_n = \Omega^*$.

We assume that $\Omega_{2n} - (\Omega_{2n-1} \cup \partial\Omega_{2n-1})$ consists of a finite number of disjoint doubly connected domains $\{A_n^i\}_{i=1}^{k(n)}$, which we call boss rings. Let the modulus of A_n^i be $\log R/r$ when we map A_n^i conformally onto a concentric circle domain $\{z; r < |z| < R\}$ and denote it by $m(A_n^i)$.

Take a countable number of disjoint closed Jordan domains $\{D_j\}_{j=1}^{\infty}$ in $\mathbb{C}$ such that

$$D_1, ..., D_{\ell(1)} \subset \Omega_1,$$

$$D_{\ell(n)+1}, ..., D_{\ell(n+1)} \subset \Omega_{2n+1} - (\Omega_{2n} \cup \partial\Omega_{2n}),\ (n = 1, 2, ...),$$

and every component of $\Omega^* - \Omega_n$ meets $\bigcup_{j=1}^{\infty} D_j$. For every D_j $(j \leq \ell(n))$, let a doubly connected domain B_n^j in $\Omega_{2n-1} - \bigcup_{i=1}^{\ell(n)} D_i$ divide D_j and $\bigcup_{j\neq i} D_i$, which we call a lorica ring. Set $\Omega = \widehat{\mathbb{C}} - \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j)$, which we call a madreporite domain.

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Definition 1. We call Ω a *madreporite domain with long bosses and loricae* if

$$\mu_n = \min\{\alpha_n, \beta_n\} \text{ and } \sum_{n=1}^{\infty} \mu_n = \infty,$$

where α_n is the minimum modulus among moduli $\{m(A_n^i)\}_{i=1}^{k(n)}$ and β_n is the minimum modulus among moduli $\{m(B_n^i)\}_{i=1}^{\ell(n)}$.

Definition 2. We say that a subdomain D in $\widehat{\mathbb{C}}$ is a *circle domain* if every boundary component of D is either a circle or a point.

We will show that a madreporite domain with long bosses and loricae is mapped conformally onto a circle domain. A madreporite domain with long bosses and loricae leads up to an infinitely generated Schottky group and relates to a condition for the group to be classical.

Definition 3. We say that a ring domain W in $\mathbb{C}$ is *nested inside* another W' if W is contained in the bounded connected component of $\mathbb{C} - W'$.

The every boss ring A_{n+1}^j is nested inside a certain boss ring among $\{A_n^i\}_{i=1}^{k(n)}$.

We use the following famous classical result due to Ahlfors and Beurling.

Proposition 1. *Let D be a domain in $\widehat{\mathbb{C}}$. Then every univalent holomorphic map of D into $\widehat{\mathbb{C}}$ is a Möbius transformation if and only if the complement of D in $\widehat{\mathbb{C}}$ belongs to the class N_D, which is, by definition, equivalent to the condition that D belongs to the class O_{AD}, i.e., that there are no non-constant holomorphic functions on D with finite Dirichlet energy.*

In particular, if the complement E of D in $\widehat{\mathbb{C}}$ belongs to the class N_D, then E is totally disconnected and every biholomorphic self-homeomorphism of D is a Möbius transformation.

Various practical tests for a compact set to belong to N_D have been considered. See for instance [8] and [9]. We use the following formulation due to McMullen [7].

Proposition 2 (Modulus test). *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of a finite union of disjoint un-nested ring domains (of finite moduli) such that* *every component W of E_{n+1} is nested inside a component of E_n, and that every sequence of nested ring domains W_n, which is a component of E_n, satisfies*

$$\sum_{n=1}^{\infty} m(W_n) = +\infty.$$

Let E'_n be the union of all bounded connected components of $\mathbb{C} - E_n$, and set

$$E = \bigcap_{n=1}^{\infty} E'_n.$$

Then E is a totally disconnected compact set belonging to N_D.

For a madreporite domain Ω with long bosses and loricae, by this modular test, the above Ω^* belongs to O_{AD} and $\widehat{\mathbb{C}} - \Omega^*$ is totally disconnected. For a point $p \in \widehat{\mathbb{C}} - \Omega^*$, every neighborhood of p meets $\bigcup_{j=1}^{\infty} D_j$. Hence $p \in \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j)$ and $p \notin \Omega$. This shows that $\Omega \subset \Omega^*$, $\Omega = \Omega - \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j) \subset \Omega^* - \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j)$. It is clear that $\Omega = \widehat{\mathbb{C}} - \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j) \supset \Omega^* - \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j)$. Thus $\Omega = \Omega^* - \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j)$. Suppose that a point $p \in \mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j) - \bigcup_{j=1}^{\infty} D_j$ belongs to Ω^*. Then there is an Ω_{2n+1} which contains p. We see that $p \in \bigcup_{j=1}^{\ell(n+1)} D_j$. This is a contradiction. Therefore $p \notin \Omega^*$ and $\Omega = \Omega^* - \bigcup_{j=1}^{\infty} D_j$.

2 A circle domain

We are concerned with the so-called circle domain theorem of Koebe [4], which has been generalized by He and Schramm [2]. We show a circle domain theorem in the case of a domain with infinite number of boundary components. The proof is in line with the case of finite-ply connected planar domains, which is essentially the same as the original one given by Koebe [4]. See also [1] and [6]. This is a different way to that of He and Schramm in [2].

Theorem 1. *Every madreporite domain Ω with long bosses and loricae can be mapped conformally onto a circle domain.*

Furthermore, for two circle domains Ω_1 and Ω_2 that are mapped conformally onto a madreporite domain with long bosses and loricae, they are conformally equivalent if and only if there is a Möbius transformation T such that $T(\Omega_1) = \Omega_2$.

Proof. There is a conformal mapping from Ω to a domain H with horizontal slits ([8], [9]). Let h_j be the horizontal slit corresponding to ∂D_j and let $H_n(\subset H)$ be the part that is mapped conformally onto $\Omega_{2n-1} \cap \Omega$.

The two H are welded along both upper edges of h_1 and along both lower edges of h_1, and we obtain a doubled planar surface W_0. There is an anti-conformal homeomorphism τ_1 of W_0 that fixes upper edge and lower edge of h_1 pointwise. The half H of W_0 has slits $\{h_i\}_{i=2}^{\ell(n)}$ and $H^* = \tau_1(H)$ has slits $\{h_i^*\}_{i=2}^{\ell(n)}$ corresponding to $\{h_i\}_{i=2}^{\ell(n)}$. For every i $(2 \leq i \leq \ell(n))$, let $\hat{W}_{0,i}^n$ (resp. $\hat{W}_{0,i}^{n*}$) be the doubled planar surface which is constructed from the W_0 and the copy $W_{0,i}^n$ (resp. $W_{0,i}^{n*}$) of W_0 welded along h_i (resp. h_i^*) as the same fashion as above. Let τ_i (resp.τ_i^*) be the anti-conformal homeomorphism of $\hat{W}_{0,i}^n$ (resp.$\hat{W}_{0,i}^{n*}$) that fix the upper edge and the lower edge of h_i (resp. h_i^*) pointwise. The composite mapping $g_j = \tau_j \circ \tau_1$ (resp. $g_j^* = \tau_j^* \circ \tau_1$) is a conformal mapping from W_0 to $W_{0,i}^n$ (resp. $W_{0,i}^{n*}$). Let W_1^n be the planar Riemann surface

$$W_0 \cup \bigcup_{i=2}^{\ell(n)} ((W_{0,i}^n \cup h_i) \cup (W_{0,i}^{n*} \cup h_i^*)).$$

Similarly we can obtain a planar Riemann surface W_2^n from W_1^n by welding $2(2\ell(n)-3)(\ell(n)-1)$-copies of W_1^n along all slits corresponding to $\{h_i, h_i^*\}_{i=2}^{\ell(n)}$ of W_1^n as the same fashion as above. Repeating this process, we can construct a planar Riemann surface W_k^n from W_{k-1}^n and finally W_n^n. In this way W_n^n is made from many copies of H by welding along slits. Similarly let $W_n^{n\#}$ be made from many copies of H_n by welding along slits. The $W_n^{n\#}$ is a subdomain of W_n^n. The sequences of planar domains $\{W_n^n\}$ and $\{W_n^{n\#}\}$ are increasing. Finally we obtain a planar Riemann surface $W = \bigcup_{n=1}^{\infty} W_n^n$. The $\{W_n^{n\#}\}$ is a regular exhaustion of W. Every τ_j can be extended to an anti-conformal involution T_j of W which fixes h_j pointwise. Now, by the uniformization theorem due to Klein, Poicaré, and Koebe, we can regard $W_n^{n\#}$, $W_{n+1}^{(n+1)\#}$, and W as domains in $\widehat{\mathbb{C}}$, which are denoted by S_n, S_{n+1}, S. By the conditions of long bosses and loricae, there is a finite union E_n of disjoint un-nested annuli that divides ∂S_{n+1} and ∂S_n, whose component is mapped conformally onto a boss ring A_n^i or a lorica ring B_n^j. The minimum modulus of the components is μ_n. Thus, by Proposition 2, we see S belongs to O_{AD}. We have anti-conformal involutions of S corresponding to T_j and denote them by the same symbol. Since the complement of S in $\widehat{\mathbb{C}}$ belongs to N_D, every T_j should be a Möbius transformation pre-composed by the complex conjugate. The T_j fixes every point on the Jordan curve C_j in S corresponding to ∂D_j. Then C_j should be a circle in $\widehat{\mathbb{C}}$. The domain $\Omega_0(\subset S)$ corresponding to the half H of W_0 is mapped conformally onto Ω. Every C_j is a boundary component of Ω_0 and, by $\widehat{\mathbb{C}} - S \in N_D$, the other boundary component is a point, which implies the first assertion. A conformal mapping from Ω_1 to Ω_2 is extended to a domain belonging to O_{AD} as above S, hence the second assertion is clear from Proposition 1. $\square$

3 Infinitely generated Schottky group

Consider a set

$$\mathcal{C} = \{C_j, C'_j \mid j \in \mathbb{N}\}$$

of countably infinite number of pairs of simple closed curves in $\mathbb{C}$ such that not only these curves but also the interiors of them are mutually disjoint. Here, the *interior* of a simple closed curve C is the bounded connected component of $\mathbb{C} - C$. The other component, together with ∞, is called the *exterior* of C. Let D_j (resp. D'_j) be the union of C_j (resp. C'_j) and the interior of C_j (resp. C'_j), and denote $\Omega(\mathcal{C}) = \widehat{\mathbb{C}} - \mathrm{Cl}(\bigcup_{j=1}^{\infty}(D_j \cup D'_j))$.

We further assume that the exterior of C_j is mapped onto the interior of C'_j by a Möbius transformation g_j for every j.

Definition 4. Let G be the group generated by all g_j defined as above. If G is discontinuous outside a compact totally disconnected set in $\widehat{\mathbb{C}}$, then we call G an *infinitely generated Schottky group* with respect to the family $\mathcal{C}$.

Here, if all elements of $\mathcal{C}$ are circles, then we call G an infinitely generated *classical* Schottky group.

Remark 1. We use, in [10], the tameness condition and the modified Maskit condition as requirement for an infinitely generated Schottky group. When the tameness condition and the modified Maskit condition are satisfied, $\Omega(\mathcal{C})$

becomes a madreporite domain with long bosses and loricae. Here the tameness condition is the following: There is an increasing sequence $\{N_i\}_{i=1}^{\infty}$ of positive integers such that, for every $N = N_i$, there is a ring domain A_i of constant modulus $d > 0$ which separates $\{C_j, C'_j \mid j = 1, \cdots, N\}$ from $\{C_j, C'_j \mid j \geq N+1\}$ and is nested inside A_{i-1}. Also, the modified Maskit condition is the following: For every element C_j of $\mathcal{C}$, there is a ring domain B_j of constant modulus $d > 0$ such that B_j separates C_j from $\mathcal{C} - \{C_j\}$. The tameness condition clearly implies that $\mathrm{Cl}(\bigcup_{j=1}^{\infty} D_j) - \bigcup_{j=1}^{\infty} D_j$ is a single point. The A_i is a boss ring and $\{B_j\}_{j=1}^{N_i}$ are lorica rings. In this case $\alpha_i = \beta_i = \mu_i = d$ and $\sum_{i=1}^{\infty} \mu_i = \infty$.

Let Ω be a madreporite domain with long bosses and loricae that satisfies the following condition:

1. $\ell(n)$ is even and denote it $2\ell(n)^*$,

2. for j $(1 \leq j \leq \ell(n)^*)$ there is a Möbius transformation g_j which maps from outside of D_{2j-1} to inside of D_{2j}.

Then $g_j(\partial D_{2j-1}) = \partial D_{2j}$ and g_j^{-1} maps from outside of D_{2j} to inside of D_{2j-1}. Let G be the group generated by all $\{g_j\}_{j=1}^{\infty}$ and call it the *group associated to a madreporite domain Ω with long bosses and loricae.*

Theorem 2. *Let G be the group associated to a madreporite domain Ω with long bosses and loricae. Then G is an infinitely generated Schottky group with respect to $\mathcal{C} = \{\partial D_{2j-1}, \partial D_{2j}\}$.*

Proof. For a set A in $\widehat{\mathbb{C}}$, put

$$\psi_n(A) = \bigcup_{j=1}^{\ell(n)^*} (g_j(A) \cup g_j^{-1}(A)) \cup A.$$

Set

$$S_{1,n} = \psi_n(\Omega) \cup \bigcup_{j=1}^{\ell(n)} \partial D_j, \quad S_{1,n}^* = \psi_n(\Omega_{2n-1} \cap \Omega) \cup \bigcup_{j=1}^{\ell(n)} \partial D_j,$$

and

$$S_{2,n} = \psi_n(S_{1,n}), ..., S_{n,n} = \psi_n(S_{n-1,n}), \; S = \bigcup_{n=1}^{\infty} S_{n,n},$$

$$S^*_{2,n} = \psi_n(S^*_{1,n}), ..., S^*_{n,n} = \psi_n(S^*_{n-1,n}).$$

Then S is a planar domain and $\{S^*_{n,n}\}$ is a regular exhaustion. By the conditions of long bosses and loricae, there is a finite union E_n of disjoint unnested ring domains which divides $\partial S^*_{n+1,n+1}$ and $\partial S^*_{n,n}$, whose component is mapped conformally onto a boss ring A^i_n or a lorica ring B^j_n. The minimum modulus of the components is μ_n. Thus we see S belongs to O_{AD}. Therefore $\widehat{\mathbb{C}} - S$ of G is totally disconnected and G is an infinitely generated Schottky group. □

We call S the *developing domain* of Ω with respect to G.

4 Infinitely generated classical Schottky group

Following Maskit [6], we introduce a Riemann surface with a symmetry.

Definition 5. We say that a Riemann surface R is *P-symmetric* with respect to a family of disjoint simple closed curves $\mathcal{L} = \{L_j\}$ if the following conditions are satisfied;

(1) there is a family $\mathcal{G} = \{\gamma_j \mid j \in \mathbb{N}\}$ of simple closed curves such that every γ_j is freely homotopic to L_j on R and R has an anti-conformal self-homeomorphism f which fixes every γ_j pointwise.

(2) $R - \bigcup_{j=1}^{\infty} \gamma_j$ is a planar domain.

It is easy to see that all γ_j are geodesics with respect to the hyperbolic metric on R. In particular, elements of $\mathcal{G}$ are mutually disjoint. The simple closed curves $\{\gamma_j\}$ play a role of *mirrors* of R. It always has another mirror by which $R - \bigcup_{j=1}^{\infty} \gamma_j$ is a symmetric planar domain.

We call $\mathcal{G}$ *P-mirrors* and f a *P-symmetric homeomorphism* with respect to $\mathcal{L}$.

For a Riemann surface R by an infinitely generated Schottky group G, the simple curve on R corresponding to C_j is denoted by L_j for every j. Set $\mathcal{L} = \{L_j \mid j \in \mathbb{N}\}$ and call it the *Schottky marking* of R corresponding to G.

Definition 6. We say that the Schottky marked Riemann surface R is *P-symmetric* if R is P-symmetric with respect to the Schottky marking.

Proposition 3. *Let $\mathcal{C}$ satisfy the tameness condition and the modified Maskit condition. If the Schottky marked Riemann surface R is P-symmetric, then $R - \bigcup_{j=1}^{\infty} \gamma_j$ is mapped conformally onto a madreporite domain with long bosses and loricae.*

Proof. By the modified Maskit condition for $\mathcal{C}$, the hyperbolic lengths of all P-mirrors $\{\gamma_j\}$ are less than a uniform constant. This is the same for the geodesic γ_i' in R freely homotopic to the essential simple closed curve in every ring domain A_i of tameness condition. By using the collar lemmas, there are disjoint ring domains $\{\tilde{B}_j(\supset \gamma_j)\}$ and $\{\tilde{A}_i(\supset \gamma_i')\}$ with a constant modulus, and there is a regular exhaustion $\tilde{\Omega}_i$ such that $\tilde{\Omega}_{2i} - \mathrm{Cl}(\tilde{\Omega}_{2i-1})$ is a ring domain with a constant modulus. This shows that $R - \bigcup_{j=1}^{\infty} \gamma_j$ is mapped conformally onto a madreporite domain with long bosses and loricae. □

Now, we can state a theorem, which is a natural generalization of a theorem of Maskit in [6].

Theorem 3. *Let G be the group associated to a madreporite domain Ω with long bosses and loricae. Further suppose that the corresponding Schottky marked Riemann surface R is P-symmetric. Then G is classical.*

Proof. Let $\mathcal{G} = \{\gamma_j \mid j \in \mathbb{N}\}$ be P-mirrors and let f be a P-symmetric homeomorphism with respect to the Schottky marking $\mathcal{L} = \{L_j \mid j \in \mathbb{N}\}$ of R. From the construction, there exists a set

$$\Gamma = \{\tilde{\gamma}_{2j-1}, \tilde{\gamma}_{2j} \mid j \in \mathbb{N}\}$$

of countable infinite number of pairs of simple closed curves in $\widehat{\mathbb{C}}$ such that $\tilde{\gamma}_{2j-1}$ and $\tilde{\gamma}_{2j}$ are projected to γ_j on R and the exterior of $\tilde{\gamma}_{2j-1}$ is mapped by the Möbius transformation g_j onto the interior of $\tilde{\gamma}_{2j}$ for every j. Let $\tilde{D}_j$ be the closed Jordan domain whose boundary is $\tilde{\gamma}_j$ and $\tilde{\Omega} = \widehat{\mathbb{C}} - \mathrm{Cl}(\bigcup_{j=1}^{\infty} \tilde{D}_j)$. The developing domain $\tilde{S}$ of $\tilde{\Omega}$ with respect to G is the same as that of Ω with respect to G. Hence $\tilde{S}$ belongs to O_{AD}. For every j, f can be lifted to an anticonformal homeomorphism τ_j of $\tilde{S}$ which has $\tilde{\gamma}_j$ as the fixed point set. Thus τ_j is a Möbius transformation pre-composed by the complex conjugate. It follows that $\tilde{\gamma}_{2j-1}$ and hence also $\tilde{\gamma}_{2j} = g_j(\tilde{\gamma}_{2j-1})$ should be a circle. Therefore G is classical. □

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