What is an optimal embedding?

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Abstract

For two finite compact bordered Riemann surfaces of the same type, assuming the possibility of an embedding one into the other, we consider a preferable embedding. As the one way, we take a subregion which has the minimum capacity among subregions with the same type whose boundaries are homotopic to the boundary of the Riemann surface. It is given as a subregion whose boundary consists of trajectories of a quadratic holomorphic differential; hence the boundary is analytic.

1 Introduction

Let R_0 be a finite compact bordered Riemann surface of genus p with m boundary components. Suppose a marking of R_0 is specified. We assume that R_0 is not simply connected and $m \geq 1$. Take the reduced Teichmüller space of R_0;

$$T(R_0) = \{(R, g); R \text{ is a finite compact bordered Riemann surface which}$$
$$\text{is mapped by a quasiconformal mapping g from } R_0 \text{ to } R\}/\sim,$$

where (R_1, g_1) is equivalent to (R_2, g_2) if there is a conformal mapping h from R_1 onto R_2 such that $g_2^{-1} \circ h \circ g_1$ is homotopic to the identity mapping. For $R_1 = (R_1, g_1) \in T(R_0)$, set

$T(R_0; R_1) = \{R_2 = (R_2, g_2) \in T(R_0);$ there is a conformal mapping f from

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R_1 into R_2 such that $g_2^{-1} \circ f \circ g_1$ is homotopic to the identity mapping$\}$, and for $(R_2, g_2) \in T(R_0; R_1)$, set

$$CE(R_1, R_2) = \{f; f \text{ is a conformal mapping from } R_1^\circ \text{ into } R_2^\circ \text{ such that } g_2^{-1} \circ f \circ g_1 \text{ is homotopic to the identity mapping}\},$$

where R_i° denotes the interior of R_i. Let R_2' be a subregion of R_2° such that the boundary is contained in R_2° and every component of $R_2^\circ - R_2'$ is doubly connected. For R_2', consider the following curve family;

$$\Gamma(R_2^\circ, R_2') = \{\gamma; \gamma \text{ consists of a family of rectifiable closed Jordan curves each of which divides the boundary components of a component of } R_2^\circ - R_2' \text{ from others and } \gamma \text{ divides all the components}\}.$$

Denote the extremal length of $\Gamma(R_2^\circ, R_2')$ by $\lambda(\Gamma(R_2^\circ, R_2'))$, i.e.,

$$\lambda(\Gamma(R_2^\circ, R_2')) = sup_\rho\{\frac{1}{A(\rho)}; \rho \text{ is a Borel measurable conformal density such that } inf_{\gamma \in \Gamma(R_2^\circ, R_2')}\{\int_\gamma \rho(z)|dz|\} \geq 1\},$$

where $A(\rho) = \int\int_{R_2} \rho^2(x+iy)dxdy$. If $(R_i, g_i) \sim (R_i', g_i')$, there is a conformal mapping h_i such that $g_i'^{-1} \circ h_i \circ g_i$ is homotopic to the identity mapping. Note that for $f \in CE(R_1, R_2)$,

$$\lambda(\Gamma(R_2^\circ, f(R_1^\circ))) = \lambda(\Gamma(R_2'^\circ, h_2 \circ f \circ h_1^{-1}(R_1'^\circ))).$$

Put

$$B(R_1, R_2) = inf\{\lambda(\Gamma(R_2^\circ, f(R_1^\circ))); f \in CE(R_1, R_2)\},$$

where $B(R_1, R_2) = \infty$ if $CE(R_1, R_2)$ is empty. We have

Theorem. *Suppose $B(R_1, R_2) < \infty$. There is an $f_0 \in CE(R_1, R_2)$ which satisfies $\lambda(\Gamma(R_2^\circ, f_0(R_1^\circ))) = B(R_1, R_2)$. The boundary of $f_0(R_1^\circ)$ consists of trajectories of a quadratic holomorphic differential on R_2; hence the boundary is analytic.*

There is a sequence $\{f_n\} \subset CE(R_1, R_2)$ such that $\lambda(\Gamma(R_2^\circ, f_n(R_1^\circ)))$ decreases to $B(R_1, R_2)$. For a bounded analytic function F on R_2, $\{F \circ f_n\}$ is a normal family. Since $\lambda(\Gamma(R_2^\circ, f_n(R_1^\circ)))$ is bounded, $f_n(R_1^\circ)$ does not get close to the boundary ∂R_2 of R_2. Since R_1 is not simply connected and $g_2^{-1} \circ f_n \circ g_1$ is homotopic to the identity mapping, $f_n(R_1^\circ)$ can not converge to a point. We may assume that $\{f_n\}$ converges to a conformal mapping f_0 from R_1° into R_2°. Since $\lambda(\Gamma(R_2^\circ, f_0(R_1^\circ))) \leq B(R_1, R_2)$, we get $\lambda(\Gamma(R_2^\circ, f_0(R_1^\circ))) = B(R_1, R_2)$. In the next section we consider the boundary of $f_0(R_1^\circ)$.

2 Variational method

At first we note that $\lambda(\Gamma(R_2^\circ, f_0(R_1^\circ)))$ gives the capacity of $f_0(R_1^\circ)$ in R_2°. Let f be a conformal embedding of R_1° into R_2° and $H(f)$ be a harmonic function on $R_2^\circ - f(R_1^\circ)$ such that $H(f)$ takes value one on the boundary of $f(R_1)$ and vanishes on the boundary of R_2. Then

$$\lambda(\Gamma(R_2^\circ, f_0(R_1^\circ))) = \|dH(f_0)\|^2 = \int\int_{R_2^0 - \overline{f_0(R_1^0)}} dH(f_0) \bigwedge *dH(f_0).$$

Now take an infinitesimally trivial dilatation μ on R_2 whose support is contained in $R_2^\circ - \overline{f_0(R_1^\circ)}$. That is

$$\int\int_{R_2} \varphi\mu \frac{d\bar{z}}{dz} = 0$$

for φ in the space $A_2^1(\hat{R}_2)$ of anti-symmetric analytic quadratic differentials with finite L^1-norm on the double of $\hat{R}_2$ of R_2 and *supp* $\mu \subset R_2^\circ - \overline{f_0(R_1^\circ)}$. Let $R_2(t)$ be the Riemann surface with the conformal structure introduced by $t\mu$. Let

$$\|\mu\|_\infty = esssup\ |\mu| < 2.$$

Then for $0 \leq t \leq \frac{1}{4}$ there is a complex dilatation $\sigma(t) \in [t\mu]$ for which

$$\|\sigma(t)\|_\infty \leq 12t^2. \ (cf.[L]\ p.227)$$

Since the part $f_0(R_1^\circ)$ of $R_2(t)$ has the same conformal structure as that of R_2, the region $f_0(R_1^\circ)$ can be regarded as a conformal embedding in $R_2(t)$. Denote it by $f_t(R_1)$. We have the following variational formula (cf. [M]);

$$\frac{d}{dt}\|dH(f_t)\|^2 = \Re - i \int\int_{R_2} (\frac{\partial}{\partial\zeta} H(f_0))^2 \mu \zeta_z^2 dz d\bar{z},$$

where ζ is a local parameter on $R_2(t)$ which satisfies

$$\frac{\zeta_{\bar{z}}}{\zeta_z} = t\mu.$$

Particularly for $t = 0$,

$$\frac{d}{dt}\|dH(f_t)\|^2|_{t=0} = \Re - i \int\int_{R_2} (\frac{\partial}{\partial z} H(f_0))^2 \mu dz d\bar{z}.$$

Suppose

$$\frac{d}{dt}\|dH(f_t)\|^2|_{t=0} = k \neq 0.$$

Then

$$\|dH(f_t)\|^2 = \|dH(f_0)\|^2 + kt + O(t^2).$$

On the other hand, the Teichmüller distance between R_2 and $R_2(t)$ is at most $12t^2$. So we can hope that there exists another embedding f_* such that

$$\lambda(\Gamma(R_2^\circ, f_*(R_1^\circ)) < \lambda(\Gamma(R_2^\circ, f_0(R_1^\circ)).$$

Although we postpone the proof, this gives a contradiction. It follows that

$$\int\int_{R_2} (\frac{\partial}{\partial z}H(f_0))^2 \mu dz d\bar{z} = 0,$$

for μ such that support of $\mu \subset R_2^\circ - f_0(R_1^\circ)$ and

$$\int\int_{R_2} \varphi\mu\frac{d\bar{z}}{dz} = 0 \ for \ \varphi \in A_2^1(\hat{R}_2).$$

Hence $(\frac{\partial}{\partial z}H(f_0))^2 dz^2$ coincides with a $\varphi_0 \in A_2^1(\hat{R}_2)$ on $R_2^\circ - f_0(R_1^\circ)$. The function $H(f_0)$ has an analytic extension across the boundary of $\partial f_0(R_1)$. Therefore $\partial f_0(R_1)$ consists of analytic curves in R_2. We remark that the embedding is uniquely determined for φ_0. For the check of above assertion, take a closed disk K which contained in $R_2^\circ - \overline{f_0(R_1^\circ)}$. Let $\varphi_1, ..., \varphi_n$ $(n = 6p + 3m - 6)$ be a basis of $A_2^1(\hat{R}_2)$. There exist Beltrami differentials $\mu_1\frac{d\bar{z}}{dz}, ..., \mu_n\frac{d\bar{z}}{dz}$ such that
i) the support of μ_i is contained in K,
ii) $\int\int \varphi_i\mu_j\frac{d\bar{z}}{dz} = a_{ij},\ det(a_{ij}) \neq 0$.
Let R_s be the Riemann surface with the conformal structure introduced by

$$\sum_{j=1}^{n} s_j\mu_j\frac{d\bar{z}}{dz} \ on \ K, s = (s_1, ..., s_n)$$

and the same conformal structure as that of R_2 on $R_2 - K$. Then $s = (s_1, ..., s_n)$ becomes a local parameter about $R_0 = R_2$ (cf. [IT]). For a small t, there exists a $s(t) = (s_1, ..., s_n)$ such that $R_{s(t)}$ is conformally equivalent to $R_2(t)$. Let h_t be the quasiconformal mapping from R_2 to $R_2(t)$ such that

$$\frac{(h_t)_{\bar{z}}d\bar{z}}{(h_t)_z dz} = \begin{cases} t\mu\frac{d\bar{z}}{dz} & on \ R_2^\circ - f_0(R_1^\circ) \\ 0 & on \ f_0(R_1^\circ), \end{cases}$$

$f_{s(t)}$ be the quasiconformal mapping from R_2 to $R_{s(t)}$ such that

$$\frac{(f_{s(t)})_{\bar{z}} d\bar{z}}{(f_{s(t)})_z dz} = \begin{cases} \sum s_j \mu_j \frac{d\bar{z}}{dz} & on\ K \\ 0 & on\ R_2^\circ - K, \end{cases}$$

and $f_{t,s}$ be the conformal mapping from $R_2(t)$ to $R_{s(t)}$ such that the quasiconformal mapping $g_t = f_{s(t)}^{-1} \circ f_{t,s} \circ h_t$ is homotopic to the identity mapping. The Beltrami coefficient of g_t converges to 0 as t converges to 0. We can assume that $g_t \circ f_0(R_1) \bigcap K = \emptyset$. Hence g_t is conformal on $f_0(R_1^\circ)$, and $g_t \circ f_0(R_1)$ becomes an embedding from R_1 into R_2. Since the order of s depends on the order t^2, we have

$$\|dH(g_t \circ f_0)\|^2 = \|dH(f_0)\|^2 + kt + O(t^2).$$

Therefore there exists τ such that

$$\|dH(g_\tau \circ f_0)\|^2 < \|dH(f_0)\|^2.$$

This contradicts the minimal property of $\|dH(f_0)\|^2$.

Remark. We believe the uniqueness of this embedding but do not have a proof. We note a certain kind of uniqueness. Let φ_0 coincide with

$$c(\frac{dz_i}{az_i(\log b_i - \log a_i)})^2$$

on the boundary component $\{z_i; |z_i| = b_i\}$ of R_2. Then minimum value is

$$B(R_1, R_2) = \sum_i \frac{2\pi}{\log b_i - \log a_i}.$$

Take real numbers c_i such that

$$\frac{\log c_i - \log a_i}{\log b_i - \log a_i} = 1 + t.$$

The local parameter z_i is regarded as a local parameter of a neighborhood of the boundary component. For a sufficiently small t, let $R(t)$ be a Riemann surface whose boundary is given by $\{z_i; |z_i| = c_i\}$. Then $R(0) = R_2,\ R(t) \in T(R_0; R_1)$ for $t > -1$. Then φ_0 coincides with

$$c(\frac{(1+t)dz_i}{az_i(\log c_i - \log a_i)})^2$$

on the boundary component $\{z_i; |z_i| = c_i\}$ of $R(t)$. The function $f_0(R_1)$ is an embedding into $R(t)$.

For $-1 < t < 0$, suppose that there is another embedding $f_1(R_1^\circ)$ into $R(t)$ such that

$$\lambda(\Gamma(R(t)^\circ, f_1(R_1^\circ))) \leq \lambda(\Gamma(R(t)^\circ, f_0(R_1^\circ))).$$

Then

$$\begin{aligned}\lambda(\Gamma(R_2^\circ, f_1(R_1^\circ)))^{-1} &> \lambda(\Gamma(R_2^\circ, R(t)^\circ)))^{-1} + \lambda(\Gamma(R(t)^\circ, f_1(R_1^\circ)))^{-1} \\ &\geq \lambda(\Gamma(R_2^\circ, R(t)^\circ)))^{-1} + \lambda(\Gamma(R(t)^\circ, f_0(R_1^\circ)))^{-1} \\ &= \lambda(\Gamma(R_2^\circ, f_0(R_1^\circ)))^{-1}.\end{aligned}$$

Hence

$$\lambda(\Gamma(R_2^\circ, f_1(R_1^\circ))) < \lambda(\Gamma(R_2^\circ, f_0(R_1^\circ))) = B(R_1, R_2).$$

This is a contradiction. Therefore $f_0(R_1)$ is the unique embedding into $R(t)$ which attains the value $B(R_1, R_2(t))$.

Similarly, we know that $R(t)$ is considered as the unique embedding into R_2 which attains the value $B(R(t), R_2)$.

3 Example

Let R_1 and R_2 be two annuli $\{z; a_1 < |z| < b_1\}$, $\{w; a_2 < |w| < b_2\}$, $(a_2 < a_1 < b_1 < b_2)$. For $f \in CE(R_1, R_2)$, let

$$\Gamma_1(f) = \{\gamma; \gamma \text{ is a Jordan curve which divides } \{f(z); |z| = a_1\} \text{ and } \{f(z); |z| = b_1\} \; in \; f(R_1)\},$$

$$\Gamma_2 = \{\gamma; \gamma \text{ is a Jordan curve which divides } \{w; |w| = a_2\} \text{ and } \{w; |w| = b_2\} \; in \; R_2\},$$

$$\Gamma_3(f) = \{\gamma; \gamma \text{ is a Jordan curve which divides } \{f(z); |z| = a_1\} \text{ and } \{w; |w| = a_2\} \text{ in a component of } R_2 - f(R_1)\},$$

$$\Gamma_4(f) = \{\gamma; \gamma \text{ is a Jordan curve which divides } \{f(z); |z| = b_1\} \text{ and } \{w; |w| = b_2\} \text{ in a component of } R_2 - f(R_1)\},$$

and

$$\Gamma(f) = \{\gamma_3 \bigcup \gamma_4; \gamma_3 \in \Gamma_3(f), \gamma_4 \in \Gamma_4(f)\}.$$

Since $\Gamma_2 \supset \Gamma_1(f) \bigcup \Gamma_3(f) \bigcup \Gamma_4(f)$, by a property of extremal length

$$\lambda(\Gamma_2)^{-1} \geq \lambda(\Gamma_1(f))^{-1} + \lambda(\Gamma_3(f))^{-1} + \lambda(\Gamma_4(f))^{-1}.$$

We have

$$\frac{1}{2\pi}(\log \frac{b_2}{a_2} - \log \frac{b_1}{a_1}) = \lambda(\Gamma_2)^{-1} - \lambda(\Gamma_1(f))^{-1}$$
$$\geq \lambda(\Gamma_3(f))^{-1} + \lambda(\Gamma_4(f))^{-1}.$$

We remark that

$$\frac{1}{2\pi}(\log \frac{b_2}{a_2} - \log \frac{b_1}{a_1}) = \lambda(\Gamma_3(f))^{-1} + \lambda(\Gamma_4(f))^{-1},$$

iff $f(R_1)$ becomes an annulus with the same center as that of R_2. There is an $f_1 \in CE(R_1, R_2)$ such that
i) $f_1(R_1)$ becomes an annulus with the same center as that of R_2,
ii) $\lambda(\Gamma_3(f_1)) = \lambda(\Gamma_3(f))$.
Then

$$\lambda(\Gamma_3(f_1))^{-1} + \lambda(\Gamma_4(f_1))^{-1} \geq \lambda(\Gamma_3(f))^{-1} + \lambda(\Gamma_4(f))^{-1}$$

and $\lambda(\Gamma_4(f_1)) \leq \lambda(\Gamma_4(f))$. Hence we have

$$\lambda(\Gamma_3(f_1)) + \lambda(\Gamma_4(f_1)) \leq \lambda(\Gamma_3(f)) + \lambda(\Gamma_4(f)).$$

So we may consider the case that the embeddings are annuli with the same center. Let $f(R_1) = \{w; a_1' < |w| < b_1'\}$. Then

$$\lambda(\Gamma(f)) = \lambda(\Gamma_3(f)) + \lambda(\Gamma_4(f))$$

$$= 2\pi\{\frac{1}{\log a_1' - \log a_2} + \frac{1}{\log b_1 - \log b_1'}\}.$$

Put $t = b_1'/a_1', s = b_2/a_2, p = \log a_2, q = \log(b_2/t)$ and $x = \log a_1'$. We can write

$$\lambda(\Gamma(f)) = 2\pi\frac{q-p}{(x-p)(q-x)}$$

$$= \frac{2\pi(q-p)}{-(x-\frac{p+q}{2})^2 + (\frac{p-q}{2})^2} \geq \frac{8\pi}{q-p}.$$

Therefore when $x = (p+q)/2$, $\lambda(\Gamma(f))$ attains the minimum value $8\pi/(q-p)$. This condition means $a'_1/a_2 = b_2/b'_1$. Only this case attains the minimum value $B(R_1, R_2)$ of $\lambda(\Gamma)$.

Remark. In this case we refer to the quadratic differential in the statement. Let $A = \{z; a < |z| < b\}$ and $H(z) = (\log|z| - \log a)/(\log b - \log a)$. Then H is called a harmonic measure for $\{z; |z| = b\}$ on A. We have

$$\|dH\|^2 = \int\int_A dH \bigwedge *dH$$

$$= \frac{1}{(\log b - \log a)^2}\int_0^{2\pi}\int_a^b \frac{drd\theta}{r} = \frac{2\pi}{\log b - \log a}.$$

Take a complex dilatation μ and let $A(t)$ be the Riemann surface with the conformal structure induced by $t\mu$. Let H_t be the harmonic measure for the outer boundary on $A(t)$, that is, H_t is harmonic in $A(t)$ and

$$H_t = \begin{cases} 0 & \text{on the inner boundary of } A(t) \\ 1 & \text{on the outer boundary of } A(t). \end{cases}$$

Since

$$\frac{\partial H}{\partial z} = \frac{1}{2(\log b - \log a)}\frac{\partial}{\partial z}\log\frac{z\bar{z}}{a^2}$$

$$= \frac{1}{2z(\log b - \log a)},$$

we have

$$\frac{d}{dt}\|dH_t\|^2_{t=0} = \Re - i\int\int_A (\frac{\partial H}{\partial z})^2 \mu dz d\bar{z}$$

$$= \frac{1}{4(\log b - \log a)^2}\Re - i\int\int_A \frac{\mu}{z^2}dzd\bar{z}.$$

For the embedding f which attains the minimum value,

$$\{\frac{1}{\log a'_1 - \log a_2}\frac{\partial}{\partial z}\log\frac{|z|}{a_2}dz\}^2 \; on \; \{z; a_2 < |z| < a'_1\}$$

and

$$\{\frac{1}{\log b_2 - \log b'_1}\frac{\partial}{\partial z}\log\frac{|z|}{b'_1}dz\}^2 \; on \; \{z; b'_1 < |z| < b_2\}$$

coincide with a quadratic differential $c(dz/z)^2$ on the double of A, because of $a'_1/a_2 = b_2/b'_1 = \exp\sqrt{c}$. From previous theory, we know that only this case attains the minimum value.

4 Schiffer's interior variation via [IT]

Let R be a Riemann surface, (U, z) be a local coordinate about p in R; $z(p) = 0, z(U) = \{z; |z| < 2\}$ and D_ρ be the inverse image of the disk $\{z; |z| < \rho\}$. For a complex parameter ϵ, define a function from U to the complex w-plane:

$$w_\epsilon(z) = z + \frac{\epsilon}{z}.$$

Delete D_ρ, $(\frac{1}{2} < \rho < 1)$ from R and paste the image $V_{\frac{1}{\rho}}$ of $D_{\frac{1}{\rho}}$ by w_ϵ the part of $D_{\frac{1}{\rho}} - D_\rho$ such that z corresponds to $w_\epsilon(z)$. We get another Riemann surface:

$$R_\epsilon = (R - D_\rho) \bigcup V_{\frac{1}{\rho}}$$

whose conformal structure coincides with that of $R - D_\rho$ in the part $R - D_\rho$ and that of $V_{\frac{1}{\rho}}$ in the part $V_{\frac{1}{\rho}}$, particularly, in the pasted part they are consistent, because w_ϵ is conformal. Consider the following mapping from R to R_ϵ;

$$f_\epsilon(p) = \begin{cases} p & p \in R - D_1 \\ w(z(p)) = z(p) + \epsilon \bar{z}(p) & p \in \overline{D_1} \end{cases}.$$

Note that $w(z(p)) = w_\epsilon(z(p)), p \in \partial D_1$. The Beltrami coefficient μ_ϵ of f_ϵ is

$$\mu_\epsilon(p) = \begin{cases} 0 & p \in R - D_1 \\ \epsilon \frac{d\bar{z}}{dz} & p \in D_1, \end{cases}$$

hence f_ϵ becomes a quasiconformal mapping from R to R_ϵ. Now take n points $\{p_i\}_{i=1\ldots n}$ and their disjoint local neighborhoods $\{U_i, z_i\}$. For n complex parameters $\epsilon = (\epsilon_1, ..., \epsilon_n)$, we can deform R to R_ϵ by the above mentioned change of conformal structure on the part of $\bigcup U_i$ and get the quasiconformal mapping f_ϵ from R to R_ϵ. Let n be the dimension of the reduced Teichmüller space of R and $\{\varphi_i\}_{i=1,\ldots,n}$ be a basis of the space $A_2^1(\hat{R})$. Consider a mapping F from the unit ball about $0 \in \mathbf{C}^n$ to the space $B^1(R)$ of Beltrami differentials with finite supremum norm:

$$F(\epsilon) = \frac{(f_\epsilon)_{\bar{z}}}{(f_\epsilon)_z} \frac{d\bar{z}}{dz} = \begin{cases} \epsilon_i \frac{d\bar{z}}{dz} & D^i = z_i^{-1}(\{z_i; |z_i| < 1\}) \\ 0 & R - \bigcup D^i \end{cases}.$$

Then

$$\frac{\partial F}{\partial \epsilon_i} = \begin{cases} \frac{d\bar{z}}{dz} & D^i \\ 0 & R - \bigcup D^i, \end{cases}$$

so F is holomorphic (cf. [L] p.206). For a $\psi \in A_2^1(\hat{R})$,

$$\int\int_R \psi \frac{\partial F}{\partial \epsilon_i} = -2\pi i \psi(p_i),$$

where $\psi = \underline{\psi}(z_i)dz_i^2, \psi(p_i) = \underline{\psi}(0)$. We can choose points $\{p_i\}$ such that

$$det(\varphi_k(p_i)) \neq 0.$$

Then $(\frac{\partial F}{\partial \epsilon_1}, ..., \frac{\partial F}{\partial \epsilon_n})$ becomes a basis of the dual space A_2^{1*} of $A_2^1(\hat{R})$ which is regarded as the tangent space of the Teichmüller space. The function F is biholomorphic. Therefore $\epsilon = (\epsilon_1, ..., \epsilon_n)$ is regarded as a local parameter of the Teichmüller space.

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