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Function Theoretic Movement of a Deforming Surface in a Space

Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

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Abstract

In this paper we are concerned with the change of function-theoretic quantities as a surface deforms in a space. A surface in the Euclidian space is regarded as a Riemann surface by the isothermal coordinate. Hence the movement of the surface in the space shall bring a change of conformal structure. So our aim is to represent variational formulas for some function-theoretic quantities in terms of displacement of the surface.

Key Words: *Riemann surfaces; Beltrami differentials; quasiconformal mappings; variational formulas.*

1. Introduction

Let S be an orientable 2 dimensional C^3-manifold with Riemannian metric:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

We can obtain an isothermal coordinate by the solution $\zeta = \xi + i\eta$ of the following Beltrami's differential equation[2];

$$\frac{\partial \zeta}{\partial \overline{w}} = \mu \frac{\partial \zeta}{\partial w} \quad (\mu = \frac{F - i(J-G)}{F - i(J+G)}, \ J = \sqrt{EG - F^2}, \ w = u + iv)$$

The first fundamental form is written as $ds^2 = \lambda (d\xi^2 + d\eta^2)$. By this isothermal coordinate a conformal structure on S is introduced and S can be regarded as a Riemann surface.

Now let a surface S be in a space and move with a parameter t as follows; every point $p = p(u,v)$ of S in a local parametric neighborhood G moves to

$$p_t = p_t(u,v) = p(u,v) + \Sigma c^i(t;u,v)e_i(u,v),$$

where $e_1 = e_1(u,v)$ and $e_2 = e_2(u,v)$ are unit tangent vectors orthogonal to each other, and $e_3 = e_3(u,v)$ is the unit normal vector, and $c^i(t;u,v)$ have at least continuous second partial derivatives with respect to t, u, v. We assume that these $\{p_t\}$ form a C^3-manifold S_t with the first fundamental form:

$$ds_t^2 = E_t du^2 + 2F_t dudv + G_t dv^2,$$

where E_t, F_t, G_t have continuous second partial derivatives with respect to t, u, v. The S_t becomes a Riemann surface by the isothermal coordinate given from ds_t. The mapping $f_t(p) = p_t$ can be regarded as a quasiconformal mapping from S to S_t. We get a quasiconformal deformation $\{S_t\}$. Under a quasiconformal deformation we have already given variational formulas of some function-theoretic quantities[5), 6), 7)]. Function theoretic quantities on S_t are often represented by the inner products of the following form:

$$A(t) = \mathrm{Re}(\phi^t \circ f_t - \phi^0, \overline{\psi^0})_S,$$

where ϕ^t, ψ^t are meromorphic differentials with certain kind of boundary behavior[5)]. The first and second variational formulas of A(t) are given as follows:

Theorem A. ([5])

(1) $$\frac{d}{dt}A(t)\,|_{t=0} = Re\, i\iint_S \phi^0 \wedge \psi^0 \frac{\partial \upsilon}{\partial t}$$

(2) $$\frac{d^2}{dt^2}A(t)\,|_{t=0} = Re\, i\{2\iint_S \phi_t^{0,0} \wedge \psi^0 \frac{\partial \upsilon}{\partial t} + \iint_S \phi^0 \wedge \psi^0 \frac{\partial^2 \upsilon}{\partial t^2}\},$$

where υ is the Beltrami differential of f_t and

$$\phi_t^{0,0} = \frac{1}{2}\{\lim_{t\to 0}\frac{\phi^t \circ f_t - \phi^0}{t} + i^*(\lim_{t\to 0}\frac{\phi^t \circ f_t - \phi^0}{t})\}.$$

According to this theorem, we will calculate the variational formulas in above-mentioned circumstances.

2. Beltrami Coefficients

We will represent the Beltrami differential of f_t in terms of the fundamental forms. The Beltrami coefficient $\upsilon(t,)$ of f_t is

$$\upsilon(t,) = \frac{Z_{\bar\zeta}}{Z_\zeta} = \frac{Z_w w_{\bar\zeta} + Z_{\bar w}\bar w_{\bar\zeta}}{Z_w w_\zeta + Z_{\bar w}\bar w_\zeta},$$

where $Z = X + iY = f_t(w)$ is the isothermal coordinate of S_t.
From

$$1 = \zeta_\zeta = \zeta_w w_\zeta + \zeta_{\bar w}\bar w_\zeta,\ 0 = \zeta_{\bar\zeta} = \zeta_w w_{\bar\zeta} + \zeta_{\bar w}\bar w_{\bar\zeta}$$

we get

$$w_\zeta = \frac{\overline{\zeta_w}}{|\zeta_w|^2 - |\zeta_{\bar w}|^2},\ w_{\bar\zeta} = \frac{-\zeta_{\bar w}}{|\zeta_w|^2 - |\zeta_{\bar w}|^2}.$$

Let

$$\mu(t,) = \frac{F_t - i(J_t - G_t)}{F_t - i(J_t + G_t)},\ (J_t = \sqrt{E_t G_t - F_t^2}),$$

then

$$Z_{\bar w} = \mu(t,)Z_w.$$

We obtain

$$\upsilon(t,) = \frac{\zeta_w(\mu(t,)-\mu)}{\overline{\zeta_w}(1-\mu(t,)\overline{\mu})}.$$

Hence the first and second derivatives of the Beltrami coefficient $\upsilon(t,)$ are obtained as follows:

$$\upsilon_t = \frac{\partial \upsilon(t,)}{\partial t} = \frac{\zeta_w(1-|\mu|^2)}{\overline{\zeta_w}(1-\mu(t,)\overline{\mu})^2}\frac{\partial \mu(t,)}{\partial t},$$

$$\upsilon_{tt} = \frac{\partial^2 \upsilon(t,)}{\partial t^2}$$

$$= \frac{\zeta_w(1-|\mu|^2)}{\overline{\zeta_w}(1-\mu(t,)\overline{\mu})^3}\{(1-\mu(t,)\overline{\mu})\frac{\partial^2\mu(t,)}{\partial t^2}+2\overline{\mu}(\frac{\partial\mu(t,)}{\partial t})^2\}.$$

Here

$$\frac{\partial}{\partial t}\mu(t,) =$$

$$\frac{(\dot{F}_t-i(\dot{J}_t-\dot{G}_t))(F_t-i(J_t+G_t))-(F_t-i(J_t-G_t))(\dot{F}_t-i(\dot{J}_t+\dot{G}_t))}{(F_t-i(J_t+G_t))^2}$$

$$=2\frac{J_t\dot{G}_t-G_t\dot{J}_t+i(F_t\dot{G}_t-G_t\dot{F}_t)}{(F_t-i(J_t+G_t))^2},$$

$$\frac{\partial^2}{\partial t^2}\mu(t,) = 2\frac{J_t\ddot{G}_t-G_t\ddot{J}_t+i(F_t\ddot{G}_t-G_t\ddot{F}_t)}{(F_t-i(J_t+G_t))^2}$$

$$-4\frac{(J_t\dot{G}_t-G_t\dot{J}_t+i(F_t\dot{G}_t-G_t\dot{F}_t))(\dot{F}_t-i(\dot{J}_t+\dot{G}_t))}{(F_t-i(J_t+G_t))^3}$$

where we denote the notation of $\frac{\partial}{\partial t}$ by $\cdot$, $\frac{\partial^2}{\partial t^2}$ by $\cdot\cdot$.

3. The Fundamental Forms of the Surface S_t

In order to calculate the derivatives of E_t, F_t, G_t, we recall the differential geometric framework [3), 4)]. Let write

$$p_u = \frac{\partial}{\partial u}p(u,v) = a_1^1 e_1(u,v) + a_1^2 e_2(u,v),$$

$$p_v = \frac{\partial}{\partial v}p(u,v) = a_2^1 e_1(u,v) + a_2^2 e_2(u,v),$$

and set

$$\theta^i = a_1^i du + a_2^i dv \quad (i=1,2), \quad \theta^3 = 0.$$

Then

$$dp = \theta^i e_i = \sum_i \theta^i e_i \quad \text{(we use Einstein's notation).}$$

Let

$$de_i = \omega_i^j e_j = \sum_j \omega_i^j e_j \quad (\omega_i^j = g_i^j du + h_i^j dv),$$

where $\omega_i^i = 0,\ \omega_j^i = -\omega_i^j,\ \omega_i^3 = b_{ij}\theta^j = -\omega_3^i.$

Then the first and the second fundamental forms are represented as follows:

$$I = Edu^2 + 2Fdudv + Gdv^2 = dpdp = \theta^1\theta^1 + \theta^2\theta^2$$

$$= \sum_{i=1}^{2}(a_1^i)^2 du^2 + 2\sum_{i=1}^{2}(a_1^i)(a_2^i)dudv + \sum_{i=1}^{2}(a_2^i)^2 dv^2,$$

$$II = Ldu^2 + 2Mdudv + Ndv^2 = -dpde_3 = -\theta^k e_k \omega_3^i e_i = b_{ij}\theta^i\theta^j$$
$$= (a_1^i du + a_2^i dv)(g_i^3 du + h_i^3 dv)$$
$$= a_1^i g_i^3 du^2 + (a_1^i h_i^3 + a_2^i g_i^3)dudv + a_2^i h_i^3 dv^2$$

Set

$$A = \begin{bmatrix} a_1^1 a_2^1 \\ a_1^2 a_2^2 \end{bmatrix},\ B = \begin{bmatrix} b_{11}\ b_{12} \\ b_{21}\ b_{22} \end{bmatrix},\ S = \begin{bmatrix} L\ M \\ M\ N \end{bmatrix} = {}^tABA,$$

$$J = \sqrt{EG-F^2} = a_1^1 a_2^2 - a_1^2 a_2^1 = \det A > 0.$$

Then the Gaussian curvature K and the Mean curvature H are denoted by

$$K = k_1k_2 = \det B = \frac{LN - M^2}{EG - F^2},$$

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\operatorname{trace} B = \frac{EN + GL - 2FM}{2(EG - F^2)},$$

where k_1 and k_2 are principal curvatures.

Now let calculate these on S_t. Write

$$dp_t = (\theta^i + \sigma^i + c^j\omega_j^i)e_i$$

where $\sigma^i = dc^i = c_u^i du + c_v^i dv.$

Then

$$I_t = E_t du^2 + 2F_t dudv + G_t dv^2 = dp_t dp_t = \sum_{i=1}^{3}(\theta^i + \sigma^i + c^j\omega_j^i)^2$$
$$= \sum_{i=1}^{3}\{(a_1^i + c_u^i + c^j g_j^i)du + (a_2^i + c_v^i + c^j h_j^i)dv\}^2.$$

Hence

$$E_t = \sum_{i=1}^{3}(a_1^i + c_u^i + c^j g_j^i)^2,$$

$$F_t = \sum_{i=1}^{3}(a_1^i + c_u^i + c^j g_j^i)(a_2^i + c_v^i + c^j h_j^i),$$

$$G_t = \sum_{i=1}^{3}(a_2^i + c_v^i + c^j h_j^i)^2.$$

It follows that

$$\dot{E}_t = 2\sum_{i=1}^{3}(a_1^i + c_u^i + c^j g_j^i)(\dot{c}_u^i + \dot{c}^k g_k^i),$$

$$\dot{F}_t = \sum_{i=1}^{3} (a_1^i + c_u^i + c^j g_j^i)(\dot{c}_v^i + \dot{c}^k h_k^i) + (a_2^i + c_v^i + c^j h_j^i)(\dot{c}_u^i + \dot{c}^k g_k^i),$$

$$\dot{G}_t = 2\sum_{i=1}^{3} (a_2^i + c_v^i + c^j h_j^i)(\dot{c}_v^i + \dot{c}^k h_k^i),$$

$$\ddot{E}_t = 2\sum_{i=1}^{3} \{(a_1^i + c_u^i + c^j g_j^i)(\ddot{c}_u^i + \ddot{c}^k g_k^i) + (\dot{c}_u^i + \dot{c}^j g_j^i)^2\},$$

$$\ddot{F}_t = \sum_{i=1}^{3} \{(a_1^i + c_u^i + c^j g_j^i)(\ddot{c}_v^i + \ddot{c}^k h_k^i)$$
$$+ (a_2^i + c_v^i + c^j h_j^i)(\ddot{c}_u^i + \ddot{c}^k g_k^i) + 2(\dot{c}_u^i + \dot{c}^j g_j^i)(\dot{c}_v^i + \dot{c}^k h_k^i)\},$$

$$\ddot{G}_t = 2\sum_{i=1}^{3} \{(a_2^i + c_v^i + c^j h_j^i)(\ddot{c}_v^i + \ddot{c}^k h_k^i) + (\dot{c}_v^i + \dot{c}^k h_k^i)^2\}.$$

We have

$$\frac{d}{dt} I_t = 2\sum_{i=1}^{3} (\theta^i + \sigma^i + c^j \omega_j^i)(\dot{\sigma}^i + \dot{c}^k \omega_k^i)$$
$$= \sum_{i=1}^{3} \dot{E}_t du^2 + 2\sum_{i=1}^{3} \dot{F}_t du dv + \sum_{i=1}^{3} \dot{G}_t dv^2,$$

$$\frac{d^2}{dt^2} I_t = 2\sum_{i=1}^{3} (\theta^i + \sigma^i + c^j \omega_j^i)(\ddot{\sigma}^i + \ddot{c}^k \omega_k^i) + (\dot{\sigma}^i + \dot{c}^k \omega_k^i)^2\}$$
$$= \sum_{i=1}^{3} \ddot{E}_t du^2 + 2\sum_{i=1}^{3} \ddot{F}_t du dv + \sum_{i=1}^{3} \ddot{G}_t dv^2,$$

where $\dot{\sigma}^i + \dot{c}^k \omega_k^i = (\dot{c}_u^i + \dot{c}^j g_j^i)du + (\dot{c}_v^i + \dot{c}^k h_k^i)dv.$

Assume that $c^i(0;u,v) = 0$, $c_u^i(0;u,v) = c_v^i(0;u,v) = 0$. Then at $t = 0$, abbreviating 0 in the notation for the convenience,

$$E = \sum_{i=1}^{3} (a_1^i)^2, \quad F = \sum_{i=1}^{3} (a_1^i)(a_2^i), \quad G = \sum_{i=1}^{3} (a_2^i)^2,$$

$$\dot{E} = 2\sum_{i=1}^{3} a_1^i (\dot{c}_u^i + \dot{c}^k g_k^i),$$

$$\dot{F} = \sum_{i=1}^{3} \{(a_1^i (\dot{c}_v^i + \dot{c}^k h_k^i) + a_2^i (\dot{c}_u^i + \dot{c}^k g_k^i)\},$$

$$\dot{G} = 2\sum_{i=1}^{3} a_2^i (\dot{c}_v^i + \dot{c}^k h_k^i),$$

$$\dot{J} = \frac{\dot{E}G + E\dot{G} - 2F\dot{F}}{2J},$$

$$\ddot{E} = 2\sum_{i=1}^{3} \{(a_1^i (\ddot{c}_u^i + \ddot{c}^k g_k^i) + (\dot{c}_u^i + \dot{c}^j g_j^i)^2\},$$

$$\ddot{F} = \sum_{i=1}^{3} \{a_1^i (\ddot{c}_v^i + \ddot{c}^k h_k^i) + a_2^i (\ddot{c}_u^i + \ddot{c}^k g_k^i)$$
$$+ 2(\dot{c}_u^i + \dot{c}^j g_j^i)(\dot{c}_v^i + \dot{c}^k h_k^i)\},$$

$$\ddot{G} = 2\sum_{i=1}^{3} \{(a_2^i (\ddot{c}_v^i + \ddot{c}^k h_k^i) + (\dot{c}_v^i + \dot{c}^k h_k^i)^2\},$$

$$\ddot{J} = \frac{\ddot{E}G + 2\dot{E}\dot{G} + E\ddot{G} - 2(\dot{F}^2 + F\ddot{F})}{2J} - \frac{\dot{J}^2}{J}$$

Now we can choose u + iv to be an isothermal coordinate of S beforehand. So assume that $a_1^1 = a_2^2 = a$, $a_1^2 = a_2^1 = 0$ and $E = G = J = a^2$, $F = 0$. Then $\theta^1 = adu$, $\theta^2 = adv$. We have

$$\begin{aligned} &Ldu^2 + 2Mdudv + Ndv^2 \\ &= adu(g_1^3 du + h_1^3 dv) + adv(g_2^3 du + h_2^3 dv), \end{aligned}$$

and

$$ag_1^3 = L,\ ah_2^3 = N,\ a(h_1^3 + g_2^3) = 2M.$$

From

$$\begin{aligned} 0 = ddp = d(\theta^i e_i) &= (d\theta^i)e_i - \theta^i \wedge de_i \\ &= (ag_2^1 - a_v)(du\wedge dv)e_1 + (a_u - ah_1^2)(du\wedge dv)e_2 \\ &\quad - a(h_1^3 - g_2^3)(du\wedge dv)e_3, \end{aligned}$$

we obtain

$$ag_2^1 = a_v,\ a_u = ah_1^2,\ h_1^3 = g_2^3$$

Therefore

$$g_2^1 = -g_1^2 = \frac{a_v}{a},\ g_1^3 = -g_3^1 = \frac{L}{a},\ g_2^3 = -g_3^2 = \frac{M}{a},$$

$$h_2^1 = -h_1^2 = \frac{-a_u}{a},\ h_1^3 = -h_3^1 = \frac{M}{a},\ h_2^3 = -h_3^2 = \frac{N}{a}.$$

Above all, we can rewrite again as follows:

$$\dot{E} = 2(a\dot{c}_u^1 + a_v\dot{c}^2 - L\dot{c}^3),$$

$$\dot{F} = a(\dot{c}_u^2 + \dot{c}_v^1) - (a_v\dot{c}^1 + a_u\dot{c}^2 + 2M\dot{c}^3),$$

$$\dot{G} = 2(a\dot{c}_v^2 + a_u\dot{c}^1 - N\dot{c}^3),$$

$$\begin{aligned} \dot{J} &= \frac{\dot{E} + \dot{G}}{2} \\ &= a(\dot{c}_u^1 + \dot{c}_v^2) + (a_u\dot{c}^1 + a_v\dot{c}^2 - (L+N)\dot{c}^3), \end{aligned}$$

$$\begin{aligned} \ddot{E} &= 2(a\ddot{c}_u^1 + a_v\ddot{c}^2 - L\ddot{c}^3) + \frac{2}{a^2}\{(a\dot{c}_u^1 + a_v\dot{c}^2 - L\dot{c}^3)^2 \\ &\quad + (a\dot{c}_u^2 - a_v\dot{c}^1 - M\dot{c}^3)^2 + (a\dot{c}_u^3 + L\dot{c}^1 + M\dot{c}^2)^2\}, \end{aligned}$$

$$\begin{aligned} \ddot{F} &= (a\ddot{c}_v^1 + a\ddot{c}_u^2 - a_u\ddot{c}^2 - a_v\ddot{c}^1 - 2M\ddot{c}^3) \\ &\quad + \frac{2}{a^2}\{(a\dot{c}_u^1 + a_v\dot{c}^2 - L\dot{c}^3)(a\dot{c}_v^1 - a_u\dot{c}^2 - M\dot{c}^3) \\ &\quad + (a\dot{c}_u^2 - a_v\dot{c}^1 - M\dot{c}^3)(a\dot{c}_v^2 + a_u\dot{c}^1 - N\dot{c}^3) \\ &\quad + (a\dot{c}_u^3 + L\dot{c}^1 + M\dot{c}^2)(a\dot{c}_v^3 + M\dot{c}^1 + N\dot{c}^2)\}, \end{aligned}$$

$$\ddot{G} = 2(a\ddot{c}_v^2 + a_u\ddot{c}^1 - N\ddot{c}^3) + \frac{2}{a^2}\{(a\dot{c}_v^1 - a_u\dot{c}^2 - M\dot{c}^3)^2$$

$$+(a\dot{c}^2_v + a_u\dot{c}^1 - N\dot{c}^3)^2 + (a\dot{c}^3_v + M\dot{c}^1 + N\dot{c}^2)^2\},$$

$$\ddot{J} = \frac{1}{2a^2}\{a^2\ddot{E} + 2\dot{E}\dot{G} + a^2\ddot{G} - 2\dot{F}^2 - 2\dot{J}^2\}$$

$$= \frac{\ddot{E}+\ddot{G}}{2} - \frac{4\dot{F}^2 + (\dot{E}-\dot{G})^2}{4a^2}.$$

4. The Derivatives of Beltrami's Coefficient

Substituting the results in the previous section, we obtain the first derivatives of $\mu(t,)$:

$$\dot{\mu} = 2\frac{J\dot{G} - G\dot{J} + i(F\dot{G} - G\dot{F})}{(F - i(J+G))^2}$$

$$= \frac{-1}{2a^2}\{2(a\dot{c}^2_v + a_u\dot{c}^1 - N\dot{c}^3) - a(\dot{c}^1_u + \dot{c}^2_v)$$
$$-(a_u\dot{c}^1 + a_v\dot{c}^2 - (L+N)\dot{c}^3)$$
$$-i(a(\dot{c}^2_u + \dot{c}^1_v) - (a_v\dot{c}^1 + a_u\dot{c}^2 + 2M\dot{c}^3))\}$$

$$= \frac{-1}{2a^2}\{a\dot{c}^2_v - a\dot{c}^1_u + a_u\dot{c}^1 - a_v\dot{c}^2 + (L-N)\dot{c}^3$$
$$-i(a(\dot{c}^2_u + \dot{c}^1_v) - (a_v\dot{c}^1 + a_u\dot{c}^2 + 2M\dot{c}^3))\}.$$

Thus we have the following

Lemma 1.

$$\dot{\mu} = \frac{1}{2}\{(\frac{\dot{c}^1}{a})_u - (\frac{\dot{c}^2}{a})_v + i((\frac{\dot{c}^2}{a})_u + (\frac{\dot{c}^1}{a})_v) + \frac{\dot{c}^3}{a^2}((N-L) - 2iM)\}.$$

In particular, if $N = L$, $M = 0$ and $\frac{\dot{c}^1}{a} + i\frac{\dot{c}^2}{a}$ is holomorphic, then $\dot{\mu}$ vanishes.

Next let calculate the second derivative of μ:

$$\ddot{\mu} = \frac{2}{(F - i(J+G))^3}\{(J\ddot{G} - G\ddot{J} + i(F\ddot{G} - G\ddot{F}))(F - i(J+G))$$
$$-2(J\dot{G} - G\dot{J} + i(F\dot{G} - G\dot{F}))(\dot{F} - i(\dot{J} + \dot{G}))\}$$

$$= \frac{1}{4ia^6}\{-2ia^4(\ddot{G} - \ddot{J} - i\ddot{F}) - 2a^2(\dot{G} - \dot{J} - i\dot{F})(\dot{F} - i(\dot{J} + \dot{G}))\}$$

$$= \frac{1}{2ia^4}\{-a^2\ddot{F} + 2\dot{F}\dot{J} - i(a^2\ddot{G} - a^2\ddot{J} - (\dot{F}^2 + \dot{G}^2 - \dot{J}^2))\}.$$

Here

$$a^2\ddot{F} - 2\dot{F}\dot{J} = a^2(a\ddot{c}^1_v + a\ddot{c}^2_u - a_u\ddot{c}^2 - a_v\ddot{c}^1 - 2M\ddot{c}^3)$$

$$
\begin{aligned}
&+2\{(a\dot c^1_u+a_v\dot c^2-L\dot c^3)(a\dot c^1_v-a_u\dot c^2-M\dot c^3)\\
&+(a\dot c^2_u-a_v\dot c^1-M\dot c^3)(a\dot c^2_v+a_u\dot c^1-N\dot c^3)\\
&+(a\dot c^3_u+L\dot c^1+M\dot c^2)(a\dot c^3_v+M\dot c^1+N\dot c^2)\}\\
&-2\{a(\dot c^2_u+\dot c^1_v)-(a_v\dot c^1+a_u\dot c^2+2M\dot c^3)\}\\
&\qquad\times\{a(\dot c^1_u+\dot c^2_v)+a_u\dot c^1+a_v\dot c^2-(L+N)\dot c^3\}\\
&=a^2(a\ddot c^1_v+a\ddot c^2_u-a_u\ddot c^2-a_v\ddot c^1-2M\ddot c^3)\\
&-2\,[(L+N)M(\dot c^3)^2-\{L(a\dot c^2_u-a_v\dot c^1)+N(a\dot c^1_v-a_u\dot c^2)\\
&+M(a\dot c^1_u+a\dot c^2_v+a_u\dot c^1+a_v\dot c^2)\}\dot c^3\\
&+(a\dot c^2_v+a_u\dot c^1)(a\dot c^1_v-a_u\dot c^2)+(a\dot c^1_u+a_v\dot c^2)(a\dot c^2_u-a_v\dot c^1)\\
&-(L\dot c^1+M\dot c^2)(M\dot c^1+N\dot c^2)-a\dot c^3_u(M\dot c^1+N\dot c^2)\\
&-a\dot c^3_v(L\dot c^1+M\dot c^2)-a^2\dot c^3_u\dot c^3_v].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&a^2\ddot G-a^2\ddot J-(\dot F^2+\dot G^2-\dot J^2)\\
&=a^2\ddot G-a^2\frac{\ddot E+\ddot G}{2}+\frac{4\dot F^2+(\dot E-\dot G)^2}{4}\\
&\qquad-(\dot F^2+\dot G^2)+\frac{(\dot E+\dot G)^2}{4}\\
&=-a^2\frac{\ddot E-\ddot G}{2}+\frac{\dot E^2-\dot G^2}{2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{\ddot E-\ddot G}{2}\\
&=(a\ddot c^1_u+a_v\ddot c^2-L\ddot c^3)+\frac{1}{a^2}\{(a\dot c^1_u+a_v\dot c^2-L\dot c^3)^2\\
&\qquad+(a\dot c^2_u-a_v\dot c^1-M\dot c^3)^2+(a\dot c^3_u+L\dot c^1+M\dot c^2)^2\}\\
&-(a\ddot c^2_v+a_u\ddot c^1-N\ddot c^3)-\frac{1}{a^2}\{(a\dot c^1_v-a_u\dot c^2-M\dot c^3)^2\\
&\qquad+(a\dot c^2_v+a_u\dot c^1-N\dot c^3)^2+(a\dot c^3_v+M\dot c^1+N\dot c^2)^2\}\\
&=a(\ddot c^1_u-\ddot c^2_v)+a_v\ddot c^2-a_u\ddot c^1-(L-N)\ddot c^3\\
&\quad+\frac{1}{a^2}[(L^2-N^2)(\dot c^3)^2+2\{M(a\dot c^1_v-a\dot c^2_u+a_v\dot c^1-a_u\dot c^2)\\
&\qquad-L(a\dot c^1_u+a_v\dot c^2)+N(a\dot c^2_v+a_u\dot c^1)\}\dot c^3\\
&\quad+a^2\{(\dot c^1_u)^2+(\dot c^2_u)^2-(\dot c^1_v)^2-(\dot c^2_v)^2+(\dot c^3_u)^2-(\dot c^3_v)^2\}\\
&\quad+2a\{a_v(\dot c^1_u\dot c^2-\dot c^2_u\dot c^1)+a_u(\dot c^1_v\dot c^2-\dot c^2_v\dot c^1)\}\\
&\quad+2a\{\dot c^3_u(L\dot c^1+M\dot c^2)-\dot c^3_v(M\dot c^1+N\dot c^2)\}\\
&\quad+((a_v)^2-(a_u)^2)((\dot c^1)^2+(\dot c^2)^2)\\
&\quad+(L^2-M^2)(\dot c^1)^2+(M^2-N^2)(\dot c^2)^2+2M(L-N)\dot c^1\dot c^2],
\end{aligned}
$$

$$\frac{\dot E^2-\dot G^2}{2}$$

$$
\begin{aligned}
&=2\,[(L^2-N^2)(\dot{c}^3)^2-2\{L(a\dot{c}^1_u+a_v\dot{c}^2)-N(a\dot{c}^2_v+a_u\dot{c}^1)\}\dot{c}^3\\
&\quad -a^2\{(\dot{c}^2_v)^2-(\dot{c}^1_u)^2\}-2a\{a_u\dot{c}^2_v\dot{c}^1-a_v\dot{c}^1_u\dot{c}^2\}\\
&\quad -(a_u)^2(\dot{c}^1)^2+(a_v)^2(\dot{c}^2)^2].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&a^2(\ddot{J}-\ddot{G})+(\dot{F}^2+\dot{G}^2-\dot{J}^2)\\
&=a^3(\ddot{c}^1_u-\ddot{c}^2_v)+a^2(a_v\ddot{c}^2-a_u\ddot{c}^1)-a^2(L-N)\ddot{c}^3\\
&\quad -(L^2-N^2)(\dot{c}^3)^2+2\{M(a\dot{c}^1_v-a\dot{c}^2_u+a_v\dot{c}^1-a_u\dot{c}^2)\\
&\qquad +L(a\dot{c}^1_u+a_v\dot{c}^2)-N(a\dot{c}^2_v+a_u\dot{c}^1)\}\dot{c}^3\\
&\quad +a^2\{-(\dot{c}^1_u)^2-(\dot{c}^1_v)^2+(\dot{c}^2_u)^2+(\dot{c}^2_v)^2+(\dot{c}^3_u)^2-(\dot{c}^3_v)^2\}\\
&\quad +2a\{a_u(\dot{c}^1_v\dot{c}^2+\dot{c}^2_v\dot{c}^1)-a_v(\dot{c}^1_u\dot{c}^2+\dot{c}^2_u\dot{c}^1)\}\\
&\quad +2a\{L\dot{c}^3_u\dot{c}^1-N\dot{c}^3_v\dot{c}^2+M(\dot{c}^3_u\dot{c}^2-\dot{c}^3_v\dot{c}^1)\}\\
&\quad +((a_u)^2+(a_v)^2)((\dot{c}^1)^2-(\dot{c}^2)^2)\\
&\quad +(L^2-M^2)(\dot{c}^1)^2+(M^2-N^2)(\dot{c}^2)^2+2M(L-N)\dot{c}^1\dot{c}^2.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
2a^4\ddot{\mu}&=(a^2(\ddot{J}-\ddot{G})+(\dot{F}^2+\dot{G}^2-\dot{J}^2)+i(a^2\ddot{F}-2\dot{F}\dot{J})\\
&=a^3(\ddot{c}^1_u-\ddot{c}^2_v)+a^2(a_v\ddot{c}^2-a_u\ddot{c}^1)-a^2(L-N)\ddot{c}^3\\
&\quad -(L^2-N^2)(\dot{c}^3)^2+2\{M(a\dot{c}^1_v-a\dot{c}^2_u+a_v\dot{c}^1-a_u\dot{c}^2)\\
&\qquad +L(a\dot{c}^1_u+a_v\dot{c}^2)-N(a\dot{c}^2_v+a_u\dot{c}^1)\}\dot{c}^3\\
&\quad +a^2\{-(\dot{c}^1_u)^2-(\dot{c}^1_v)^2+(\dot{c}^2_u)^2+(\dot{c}^2_v)^2+(\dot{c}^3_u)^2-(\dot{c}^3_v)^2\}\\
&\quad +2a\{a_u(\dot{c}^1_v\dot{c}^2+\dot{c}^2_v\dot{c}^1)-a_v(\dot{c}^1_u\dot{c}^2+\dot{c}^2_u\dot{c}^1)\}\\
&\quad +2a\{L\dot{c}^3_u\dot{c}^1-N\dot{c}^3_v\dot{c}^2+M(\dot{c}^3_u\dot{c}^2-\dot{c}^3_v\dot{c}^1)\}\\
&\quad +((a_u)^2+(a_v)^2)((\dot{c}^1)^2-(\dot{c}^2)^2)\\
&\quad +(L^2-M^2)(\dot{c}^1)^2+(M^2-N^2)(\dot{c}^2)^2+2M(L-N)\dot{c}^1\dot{c}^2\\
&\quad +i\{a^2(a\ddot{c}^1_v+a\ddot{c}^2_u-a_u\ddot{c}^2-a_v\ddot{c}^1-2M\ddot{c}^3)\\
&\quad -2\,[(L+N)M(\dot{c}^3)^2-\{L(a\dot{c}^2_u-a_v\dot{c}^1)+N(a\dot{c}^1_v-a_u\dot{c}^2)\\
&\quad +M(a\dot{c}^1_u+a\dot{c}^2_v+a_u\dot{c}^1+a_v\dot{c}^2)\}\dot{c}^3\\
&\quad +(a\dot{c}^2_v+a_u\dot{c}^1)(a\dot{c}^1_v-a_u\dot{c}^2)+(a\dot{c}^1_u+a_v\dot{c}^2)(a\dot{c}^2_u-a_v\dot{c}^1)\\
&\quad -(L\dot{c}^1+M\dot{c}^2)(M\dot{c}^1+N\dot{c}^2)-a\dot{c}^3_u(M\dot{c}^1+N\dot{c}^2)\\
&\quad -a\dot{c}^3_v(L\dot{c}^1+M\dot{c}^2)-a^2\dot{c}^3_u\dot{c}^3_v]\}\\
&=a^3(\ddot{c}^1_u-\ddot{c}^2_v+i(\ddot{c}^1_v+\ddot{c}^2_u))\\
&\quad -a^2(a_u\ddot{c}^1-a_v\ddot{c}^2+i(a_u\ddot{c}^2+a_v\ddot{c}^1)\\
&\quad -a^2(L-N+2iM)\ddot{c}^3\\
&\quad -(L+N)(L-N+2iM)(\dot{c}^3)^2\\
&\quad +2\{M(a\dot{c}^1_v-a\dot{c}^2_u+a_v\dot{c}^1-a_u\dot{c}^2+i(a\dot{c}^1_u+a\dot{c}^2_v+a_u\dot{c}^1+a_v\dot{c}^2)\\
&\quad +L(a\dot{c}^1_u+a_v\dot{c}^2+i(a\dot{c}^2_u-a_v\dot{c}^1))-N(a\dot{c}^2_v+a_u\dot{c}^1-i(a\dot{c}^1_v-a_u\dot{c}^2))\}\dot{c}^3\\
&\quad -a^2\{(\dot{c}^1_u)^2-(\dot{c}^2_u)^2+2i\dot{c}^1_u\dot{c}^2_u+(\dot{c}^1_v)^2-(\dot{c}^2_v)^2+2i\dot{c}^1_v\dot{c}^2_v\\
&\qquad -(\dot{c}^3_u)^2+(\dot{c}^3_v)^2-2i\dot{c}^3_u\dot{c}^3_v\}\\
&\quad +2a\{a_u(\dot{c}^1_v\dot{c}^2+\dot{c}^2_v\dot{c}^1-i(\dot{c}^1_v\dot{c}^1-\dot{c}^2_v\dot{c}^2))\\
&\qquad -a_v(\dot{c}^1_u\dot{c}^2+\dot{c}^2_u\dot{c}^1-i(\dot{c}^1_u\dot{c}^1-\dot{c}^2_u\dot{c}^2))\}\\
&\quad +2a\{L(\dot{c}^3_u+i\dot{c}^3_v)\dot{c}^1+iN(\dot{c}^3_u+i\dot{c}^3_v)\dot{c}^2
\end{aligned}
$$

$$+iM((\dot{c}^3_u + i\dot{c}^3_v)(\dot{c}^1 - i\dot{c}^2)\}$$
$$+((a_u)^2 + (a_v)^2)((\dot{c}^1)^2 - (\dot{c}^2)^2 + 2i\dot{c}^1\dot{c}^2)$$
$$+(L^2 - M^2 + 2iLM)(\dot{c}^1)^2 + (M^2 - N^2 + 2iMN)(\dot{c}^2)^2$$
$$+2(M(L-N) + i(LN+M^2))\dot{c}^1\dot{c}^2.$$

Put this in order and we have the following

Lemma 2.

$$\begin{aligned}\ddot{\mu} = \frac{1}{2a^4}\, [& a^3((\ddot{c}^1_u + i\ddot{c}^2_u) + i(\ddot{c}^1_v + i\ddot{c}^2_v)) \\ & -a^2(\ddot{c}^1 + i\ddot{c}^2)(a_u + ia_v) \\ & -a^2(L-N+2iM)\ddot{c}^3 - (L+N)(L-N+2iM)(\dot{c}^3)^2 \\ & +2\{iM(a(\dot{c}^1_u + i\dot{c}^2_u - i(\dot{c}^1_v + i\dot{c}^2_v)) + (a_u - ia_v)(\dot{c}^1 + i\dot{c}^2)) \\ & +L(a(\dot{c}^1_u + i\dot{c}^2_u) - ia_v(\dot{c}^1 + i\dot{c}^2)) + iN(a(\dot{c}^1_v + i\dot{c}^2_v) + ia_u(\dot{c}^1 + i\dot{c}^2)\}\dot{c}^3 \\ & -a^2\{(\dot{c}^1_u + i\dot{c}^2_u)^2 + (\dot{c}^1_v + i\dot{c}^2_v)^2 - (\dot{c}^3_u + i\dot{c}^3_v)^2\} \\ & +2a\{-ia_u(\dot{c}^1_v + i\dot{c}^2_v)(\dot{c}^1 + i\dot{c}^2)) \\ & \quad +ia_v(\dot{c}^1_u + i\dot{c}^2_u)(\dot{c}^1 + i\dot{c}^2)) \\ & +2a\{L(\dot{c}^3_u + i\dot{c}^3_v)\dot{c}^1 + iN(\dot{c}^3_u + i\dot{c}^3_v)\dot{c}^2 \\ & \quad +iM((\dot{c}^3_u + i\dot{c}^3_v)(\dot{c}^1 - i\dot{c}^2)\} \\ & +((a_u)^2 + (a_v)^2)(\dot{c}^1 + i\dot{c}^2)^2 \\ & +(L+iM)^2(\dot{c}^1)^2 + (M+iN)^2(\dot{c}^2)^2 \\ & +2(M(L-N) + i(LN+M^2))\dot{c}^1\dot{c}^2].\end{aligned}$$

In particular, if $c^1 \equiv c^2 \equiv 0$, *then*

$$\ddot{\mu} = \frac{1}{2a^4}\, [a^2(\dot{c}^3_u + i\dot{c}^3_v)^2 - (L-N+2iM)\{a^2\ddot{c}^3 + (L+N)(\dot{c}^3)^2\}].$$

6. Variational Formulas

By Lemma 1 and 2 we have

Proposition.

$$\begin{aligned}\frac{\partial \nu}{\partial t}\Big|_{t=0} = \frac{1}{2}\Big\{ & \frac{\partial^2}{\partial u\,\partial t}\Big(\frac{c^1}{\sqrt{J}}\Big) - \frac{\partial^2}{\partial v\,\partial t}\Big(\frac{c^2}{\sqrt{J}}\Big) \\ & +i\Big[\frac{\partial^2}{\partial u\,\partial t}\Big(\frac{c^2}{\sqrt{J}}\Big) + \frac{\partial^2}{\partial v\,\partial t}\Big(\frac{c^1}{\sqrt{J}}\Big)\Big] - \frac{L-N+2iM}{J}\frac{\partial c^3}{\partial t}\Big\}\frac{\overline{dw}}{dw}.\end{aligned}$$

If $c^1 \equiv c^2 \equiv 0$, *then*

$$\begin{aligned}\frac{\partial^2 \nu}{\partial t^2}\Big|_{t=0} = \frac{1}{2J^2}\Big\{ & J\Big(\frac{\partial^2 c^3}{\partial u\,\partial t} + i\frac{\partial^2 c^3}{\partial v\,\partial t}\Big)^2 - J(L-N+2iM)\frac{\partial^2 c^3}{\partial t^2} \\ & -(L+N)(L-N+2iM)\Big(\frac{\partial c^3}{\partial t}\Big)^2\Big\}\frac{\overline{dw}}{dw}.\end{aligned}$$

Applying this proposition to theorem A, we obtain the following

Theorem. *If* $c^1 \equiv c^2 \equiv 0$ *and* $L = N$, $M = 0$, *then*

$$\frac{\partial v}{\partial t}\Big|_{t=0} = 0, \quad \frac{\partial^2 v}{\partial t^2}\Big|_{t=0} = \frac{1}{2J}\left(\frac{\partial^2 c^3}{\partial u\,\partial t} + i\,\frac{\partial^2 c^3}{\partial v\,\partial t}\right)^2 \frac{\overline{dw}}{dw}$$

and

$$\frac{d}{dt}A(t)\Big|_{t=0} = 0,$$

$$\frac{d^2}{dt^2}A(t)\Big|_{t=0} = Re\, i\iint_G \frac{1}{2J}\left(\frac{\partial^2 c^3}{\partial u\,\partial t} + i\,\frac{\partial^2 c^3}{\partial v\,\partial t}\right)^2 \phi^0 \wedge \phi^0\, \frac{\overline{dw}}{dw}\,.$$

When the surface deforms to the normal direction in the part G contained in a plane or spherical surface, the state of variation of function theoretic quantities depends on the sign of

$$\mathrm{Re}\, i\iint_G \frac{1}{2J}\left(\frac{\partial^2 c^3}{\partial u\,\partial t} + i\,\frac{\partial^2 c^3}{\partial v\,\partial t}\right)^2 \phi^0 \wedge \phi^0\, \frac{\overline{dw}}{dw}\,.$$

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