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*Abstract*—**A new sufficient condition for a polytope of matrices to be Hurwitz-stable is presented. The stability is a consequence of the existence of a parameter-dependent quadratic Lyapunov function, which is assured by a certain linear constraint for generating extreme matrices of the polytope. The condition can be regarded as a duality of the known extreme point result on quadratic stability of matrix polytopes, where a fixed quadratic Lyapunov function plays the role. The obtained results are applied to a polytope of second-degree polynomials for illustration.**

*Index Terms*—**Hurwitz-stability, parameter-dependent Lyapunov function, polytope of matrices, quadratic stability.**

## I. Introduction

Polytopes of matrices are now established as one of standard representations of uncertainties involved in state-space models of control systems [1], [2]. When the system matrices of uncertain systems are formulated by a polytope of matrices, a stability problem of the polytope naturally arises. It is known that one generally cannot expect the extreme point result on stability, Hurwitz or Schur alike, of polytopes of matrices. That is, stability of the generating extreme matrices does not necessarily imply that of every matrix in the polytope. This means that in order to assure stability of a polytope we have to impose additional constraints to the stability condition or stricter conditions than that for each extreme matrix. Considerable numbers of such conditions are currently in hand (see, e.g., the references in [3]), but each of them has its own demerit. For example, diagonal dominance-type conditions for Hurwitz stability require negativity of diagonal entries of the extreme matrices, an apparent restriction to their applicability.

This paper presents a new sufficient condition for Hurwitz-stability of a polytope of matrices, thus providing an alternative to the existing tools. The stability comes from a parameter-dependent quadratic Lyapunov function, the existence of which is ensured by a linear constraint for the extreme matrices. The obtained condition can be considered as a kind of dual of the quadratic stability result on a polytope of matrices, where, by contrast, a fixed quadratic Lyapunov function plays the role. As an illustrative example, we look into the Lyapunov function problems of a polytope of polynomials, which are connected to the matrix counterpart with a companion form. The contents of the paper are laid out as follows. In the next section, the main result is stated along with the quadratic stability result, which is known, but can also be proved in the context of the present approach. Section III includes an application of the results to a Lyapunov function problem for a polytope of second-degree polynomials. Discussions are given on which type of Lyapunov functions can cover the polytopes of the polynomials in the coefficient plane. Some comments are also made on the differences between the present results and existing analysis methods that utilize parameter-dependent Lyapunov functions. Finally, inSection IV, several remarks are given to conclude the paper.

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## II. Main Results

Throughout the paper, we identify a linear constant continuous-time system $\dot{x} = Ax$ and a quadratic Lyapunov function $x'Px$, ($'$): transpose with their coefficient matrices $A$ and $P$, respectively. We say $P$ covers $A$, when $P$ is a Lyapunov function that guarantees Hurwitz-stability of the system $A$. Let $\Pi_\alpha$ be a set of $m$-tuple of nonnegative numbers defined by

$$\Pi_\alpha := \left\{ \alpha = (\alpha_1, \cdots, \alpha_m) \,\middle|\, \sum_{i=1}^{m} \alpha_i = 1, \alpha_i \geq 0, i \in \underline{m} \right\}$$

where $\underline{m} := \{1, \cdots, m\}$. For given $m$ matrices, $X_i \in R^{n\times n}$, $i \in \underline{m}$, we define a polytope of the matrices by

$$\mathcal{P}(X_i) := \left\{ X_\alpha \,\middle|\, X_\alpha = \sum_{i=1}^{m} \alpha_i X_i, \alpha \in \Pi_\alpha \right\}.$$

Now, assume we are given a set of $m$ Hurwitz-stable matrices $\{A_i\}$ where $A_i \in R^{n\times n}$, $i \in \underline{m}$. Our concern is to find an additional condition for ensuring Hurwitz-stability of the polytope of matrices $\mathcal{P}(A_i)$. The following theorem is the main contribution of this paper.

*Theorem 1:* Suppose $A_i$, $i \in \underline{m}$ are Hurwitz-stable. If there exist $S = -S' \in R^{n\times n}$ and $Q = Q' \in R^{n\times n}$ with $Q > 0$ (positive definite) satisfying

$$A_i'(S - Q) + (S + Q)A_i = 0, \qquad i \in \underline{m} \tag{1}$$

then any member $A_\alpha$ in the polytope $\mathcal{P}(A_i)$ is Hurwitz-stable.

The stability follows from the fact that (1) guarantees the existence of the parameter-dependent quadratic Lyapunov function of the form

$$P_\alpha = \left( \sum_{i=1}^{m} \alpha_i P_i^{-1} \right)^{-1} \tag{2}$$

which covers $A_\alpha$. In (2), $P_i$ are the solutions to the Lyapunov equations

$$A_i'P_i + P_iA_i = -2Q, \qquad i \in \underline{m}. \tag{3}$$

For the proof, the following lemma is crucial.

*Lemma [4]:* If a matrix $A$ is Hurwitz-stable, then for any $Q = Q' > 0$ it can be written as

$$A = P^{-1}(S - Q) \tag{4}$$

where $P = P' > 0$ is the solution to the Lyapunov equation

$$A'P + PA = -2Q \tag{5}$$

and $S = -S'$ is given by

$$S = \tfrac{1}{2}(PA - A'P). \tag{6}$$

*Remark 1:* The expression (4) states that the symmetric part of $A'P$ is $-Q$ and the antisymmetric one $-S$. Furthermore, by eliminating $P$ from (5) and (6), we obtain

$$A'S + SA = A'Q - QA. \tag{7}$$

Equations (5) and (7) show that once $Q$ is given, $P$ and $S$ can be determined uniquely through them, respectively. Thus, (4) has the freedom of choosing an arbitrary positive definite matrix $Q$.

We are now able to give the proof of the theorem.

*Proof of Theorem 1:* Since (1) can be written as

$$A_i'S + SA_i = A_i'Q - QA_i, \qquad i \in \underline{m} \tag{8}$$

the lemma underscores that $A_i$ can be expressed as

$$A_i = P_i^{-1}(S - Q), \qquad i \in \underline{m}. \tag{9}$$

With these relations, we see that any $A_\alpha \in \mathcal{P}(A_i)$ can take the form of

$$A_\alpha = \sum_{i=1}^{m} \alpha_i A_i = \sum_{i=1}^{m} \alpha_i P_i^{-1}(S - Q) = P_\alpha^{-1}(S - Q) \tag{10}$$

where

$$P_\alpha = \left( \sum_{i=1}^{m} \alpha_i P_i^{-1} \right)^{-1}. \tag{11}$$

Because $P_\alpha = P_\alpha' > 0$, resorting again to the lemma, we can confirm that $A_\alpha$ is Hurwitz-stable and $P_\alpha$ satisfies the Lyapunov equation

$$A_\alpha'P_\alpha + P_\alpha A_\alpha = -2Q. \tag{12}$$

This shows the proof is complete. Q.E.D.

*Remark 2:* Condition (1) is indeed a necessary and sufficient condition for the existence of the Lyapunov function of the form (2) where $P_i$ given by (3) satisfies $P_iA_i - A_i'P_i = 2S$, $i \in \underline{m}$. To figure this out, note that the sufficiency part is just the foregoing proof. On the other hand, the lemma and Remark 1 indicate that the conditions imposed upon $P_i$ result in (1), giving the necessity part. It thus turns out that (1) is an exact existence condition for the parameter-dependent Lyapunov function.

Note that (8) requires that $P_iA_i$, $i \in \underline{m}$ are all equal. This additional constraint enables us to obtain an exact existence condition of the specific Lyapunov function of the form (11) as remarked above.

When $Q$ and $S$ are given, the condition (1) of the theorem specifies a linear set in $n^2$-dimensional matrix entry space. The set, denoted by $\Gamma(Q, S)$, includes the given polyope of matrices. With the constraints, $Q = Q' > 0$ and $S = -S'$, (1) gives a family of parameterized linear sets $\cup_{Q,S}\Gamma(Q, S)$ in the entry space of the system matrix $A$. Every polytope within the intersection of any one of the sets and of the Hurwitz-stability regions in the entry space has a parameter-dependent Lyapunov function. An important observation is that the whole Hurwitz-stability regions in the entry space are also covered by families of the set, $\Gamma(Q, S)$.

The theorem owes its result to the linearity of the representation of a Hurwitz matrix $A$ with respect to $P^{-1}$ for a given $S$ and $Q$ as in (4). If the setting of these matrices is interchanged, i.e., with $P$ being fixed, we arrive at another stability condition, which turns out to be a well-known extreme point result on quadratic stability of polytopes of matrices. We are reminded that a set of systems is said to be quadratically stable, if there exists a fixed quadratic Lyapunov function which covers the systems [10].

*Quadratic Stability Condition for Matrix Polytopes* (see [2, pp. 343–346]).

A necessary and sufficient condition for a polytope $\mathcal{P}(A_i)$ to be quadratically stable is the existence of a fixed solution $P = P' > 0$ to the equations

$$A_i'P + PA_i = -2Q_i, \qquad i \in \underline{m} \tag{13}$$

where $Q_i = Q_i'$ are some positive definite matrices. In other words, (13) is the quadratic stability conditions for the set of the extreme matrices. Under these conditions, any $A_\alpha \in \mathcal{P}(A_i)$ can also be represented by

$$A_\alpha = P^{-1}(S_\alpha - Q_\alpha) \tag{14}$$

where $Q_\alpha = \sum_{i=1}^{m} \alpha_i Q_i \in \mathcal{P}(Q_i)$, $S_\alpha = \sum_{i=1}^{m} \alpha_i S_i \in \mathcal{P}(S_i)$, and $S_i$ are solutions to (8) with $Q$ replaced by $Q_i$.

Notice that exact conditions are also obtained by strengthening the stability concept. Because both the outcomes of the theorem and the above result originate from the bilinearity of the same matrix expression (4), each of them can be regarded as a kind of dual of the other with respect to (4). The quadratic stability condition boils down to the problem of checking the existence of a common quadratic Lyapunov function, a typical common Lyapunov function problem. Although a complete analytical characterization of the set $\{A_i\}$ having a Lyapunov function is nonexistent, several its subclasses are found [6].

## III. Illustrative Example and Comments

In this section, the previous results are applied for illustration to a Lyapunov function problem of a polytope of polynomials and some comments are given on the results in connection with existing analysis tools that use parameter-dependent Lyapunov functions. We first consider the Lyapunov function problem of a polytope of polynomials. Polynomials are linked here to quadratic Lyapunov functions through a corresponding companion form. Then the Lyapunov functions are obtained as solutions to the Lyapunov inequalities with the coefficients in the companion form. For example, a second-degree polynomial, $s^2 + a_2 s + a_1$, is related to the companion form

$$A_c = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \tag{15}$$

and this gives a quadratic Lyapunov function. Observe that through the companion form polytopes of matrices have a one-to-one correspondence with those of polynomials. Our concern is not the stability of polytopes of polynomials but the existence of certain quadratic Lyapunov functions which cover them. For the sake of simplicity, consider the case of $n = 2$ and concentrate on the first quadrant of the coefficient plane $(a_1, a_2)$. This makes us free from the stability consideration for the polynomials. With abuse of expressions, a region in the coefficient plane is said to have a quadratic Lyapunov function (fixed or otherwise), if a set of the corresponding matrices is covered by the Lyapunov function. We are interested in the types of Lyapunov functions that cover certain polytopes of polynomials, which are associated with convex polygonal regions in $(a_1 a_2)$ plane.

Now, setting in (7)

$$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & s \\ -s & 0 \end{bmatrix}, \quad A = A_c \tag{16}$$

and solving the equation, we obtain a scalar linear relation

$$q_1 + a_1 q_2 - a_2(q_3 + s) = 0. \tag{17}$$

Since $q_1 > 0$ and $q_2 > 0$ require $q_3 + s > 0$, (17) represents a line with slant $q_2/(q_3 + s) > 0$ and intercept $q_1/(q_3 + s) > 0$ in the coefficient plane. To see the results concretely, we set $q_1 = q_2 = 1$, $q_3 = 0$, $s = 2$ in (17), giving the line $l$: $1 + a_1 - 2a_2 = 0$ in $(a_1, a_2)$ plane which passes through the points $U = U_{11}$ and $V_{32}$ (see Fig. 1). Due to the theorem, the line segment $U_{11}V_{32}$ is covered by a parameter-dependent Lyapunov function determined at the two extreme points, $U_{11}$ and $V_{32}$. In fact, in accordance with (9) we have at these points

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \left( \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$

$$\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 8/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}^{-1} \left( \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

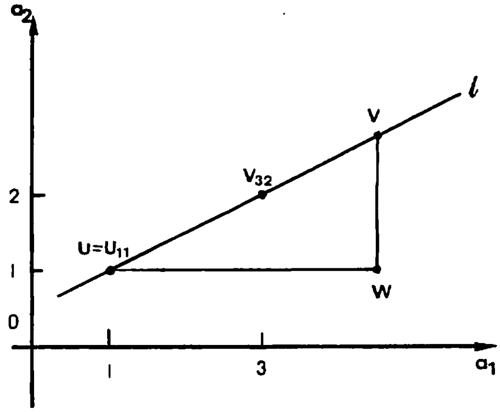


Fig. 1. Coefficient plane for 2nd-degree polynomials.

Due to the theorem, an internally dividing point of the segment $U_{11}V_{32}$ with the ratio $\alpha_1 : \alpha_2$ admits a parameter-dependent Lyapunov function

$$P_\alpha = \left( \alpha_1 \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} + \alpha_2 \begin{bmatrix} 8/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}^{-1} \right)^{-1}$$
$$\alpha = (\alpha_1, \alpha_2) \in \Pi_\alpha. \tag{18}$$

The situation is the same for any other segments (polytopes) on the affine set $l$ and any of them is covered by a certain parameter-dependent Lyapunov function determined at its extreme points. Moreover, in light of (17), we see that the whole triangle $UVW$ where $V$ is any point on $l$ can be filled with continuum of lines with the form of (17). In this sense, the triangle is densely covered by a family of parameter-dependent Lyapunov functions, whose member function corresponds to some specified values of $q_1$, $q_2$, $q_3$ and $s$.

Now, we pass to results that the quadratic stability condition yields for the triangle. They are in fact available in recent literature, which considers quadratic stability of interval polynomials [5]. The most relevant fact among them is that while any $a_2$-axis-parallel segment is quadratically stable, no $a_1$-axis counterpart can remain quadratically stable as the segment length grows. Taking account of the continuity of the quadratic stability property in the coefficient plane, things about the stability for the triangle are as follows. If we choose $V$ near enough to $U$, the triangle is quadratically stable. As $V$ moves away from $U$ on $l$, it reaches the point where the quadratic stability property can no longer be sustained. For any $V$ beyond that point on the line $l$, quadratic stability holds no more for the resulting triangle. Note, however, any segments in the first quadrant of $(a_1, a_2)$ plane are quadratically stable so long as they are $a_2$-axis-parallel. The comparison made thus far between the outcomes of the two Lyapunov functions appears to illuminate a rough contrast between their natures: global property *vs*, local one in the matrix entry space. To sum up generally, both results do not have any inclusion relations and can therefore supplement each other.

In the literature, we can find some results that analyze affinely perturbed linear systems with affine parameter-dependent Lyapunov functions [7]–[9]. When the affine system expressions are rearranged in polytopic forms, these results could provide an answer to the stability questions of a polytope of system matrices using Lyapunov equations at the generating matrices. The main difference of the theorem from the results of [8] and [9] is that the theorem gives an exact existence condition of a specific Lyapunov function as stated in Remark 2, whereas in [8] or [9] the condition is merely a sufficient one for the affine Lyapunov function. The result in [7] is exact, but one has to construct augmented Lyapunov functions including parameters at the corner matrices. Another apparent difference is the types of the Lyapunov functions: in [7]–[9] affine and in the theroem "inverse affine." As to sharpness of the condition, however, the theorem would not be able to claim superiority in general, because it stems from the same Hurwitz matrix expression as the quadratic stability result, a special case of the affine type Lyapunov functions, and also because it imposes the condition that the right-hand side of the Lyapunov equation being constant. Notwithstanding, the theorem points out the condition under which a new type of parameter-dependent Lyapunov function that covers the polyope of matrices exists.

## IV. Concluding Remarks

For a polytope of Hurwitz-stable matrices, a sufficient Hurwitz-stability condition is derived. The condition is a linear relation in the matrix entry space for each generating extreme matrix. If the conditions are met, there exists a parameter-dependent Lyapunov function which ensures the stability of the polytope. The result bears a kind of dual relationship with the established extreme point result on quadratic stability of a polytope of matrices in the sense that both come from the bilinearity of the Hurwitz matrix expression. Applications to Lyapunov function problems for polytopes of second-degree polynomials are shown for illustration. A Schur-stability counterpart possibly holds true and studies thereof are underway. Computational issues for checking the obtained condition also remain to be examined.

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