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LETTER

Relaxed Monotonic Conditions for Schur Stability of Real Polynomials

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SUMMARY Monotonic condition, a well-known sufficient condition for Schur stability of real polynomials, is relaxed. The condition reads that a series of strictly and monotonically decreasing positive coefficients of the polynomials yields Schur stability. It is shown by inspecting the original proof that equalities are allowed in all the inequalities but two which are located at appropriate positions.

key words: *monotonic condition, Schur stability, real polynomial, sufficient condition, Jury stability test*

1. Introduction

Sufficient Schur stability conditions of polynomials are still of much use, although the exact conditions of several forms are available. For example, a preliminary simple check method would be preferable to the intensive computation requirement of the exact stability tests. So far, a number of sufficient conditions for the Schur stability of real polynomials has been studied in [1], [2], [4]–[9]. Among these conditions, monotonic condition [2] is a typical one for Schur stability of a real polynomial. The condition is so simple that it can be checked at a glance. The monotonic condition also possesses another feature: it requires the positivity of all the coefficients of a polynomial. These two features, simplicity and positivity, could fit in some specific control applications, where controller and controlled variables are invariably non-negative [11]. A typical amongst them is a controller design of certain communication networks in which controlled variable is queue length, a positive number [10]. The motivation of the present work originates from that design scheme, where relaxation of the strict inequalities in the monotonic condition was strongly desired. This leads to the results of this note. It is pointed out that by looking into the original proof in [2] closely, the required strict inequalities can be relaxed in such a way that equalities are allowed in all the inequalities but two which are located at appropriate positions. These results will be proven with several comments in Sect. 3 after stating the Schur stability test by Jury as a preliminary lemma in Sect. 2. Section 4 concludes the note.

2. Preliminaries

We first state the monotonic condition and Jury stability test method given in [2] as follows:

Lemma 1: Given an n-th degree real polynomial

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-2} z^{n-2} + a_{n-1} z^{n-1} + a_n z^n,\ a_n > 0, \tag{1}$$

the monotonic condition

$$a_n > a_{n-1} > a_{n-2} > \ldots > a_2 > a_1 > a_0 > 0 \tag{2}$$

is sufficient for Schur stability of $F(z)$.

This condition was proven by Jury stability test:

Lemma 2: A necessary and sufficient condition for the stability of the polynomial $F(z)$ is that in the following Jury stability test table

Row	z^0	z^1	z^2	z^3	...	z^{n-3}	z^{n-2}	z^{n-1}	z^n
0	a_0	a_1	a_2	a_3	...	a_{n-3}	a_{n-2}	a_{n-1}	a_n
1	$a_0^{(1)}$	$a_1^{(1)}$	$a_2^{(1)}$	$a_3^{(1)}$	...	$a_{n-3}^{(1)}$	$a_{n-2}^{(1)}$	$a_{n-1}^{(1)}$	
2	$a_0^{(2)}$	$a_1^{(2)}$	$a_2^{(2)}$	$a_3^{(2)}$	...	$a_{n-3}^{(2)}$	$a_{n-2}^{(2)}$		
3	$a_0^{(3)}$	$a_1^{(3)}$	$a_2^{(3)}$	$a_3^{(3)}$	...	$a_{n-3}^{(3)}$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$					
$n-3$	$a_0^{(n-3)}$	$a_1^{(n-3)}$	$a_2^{(n-3)}$	$a_3^{(n-3)}$					
$n-2$	$a_0^{(n-2)}$	$a_1^{(n-2)}$	$a_2^{(n-2)}$						

where the row $1, 2, 3, \ldots, (n-2)$'s elements are determined successively by

$$a_k^{(1)} = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix} = a_0 a_k - a_n a_{n-k},\ k = 0, 1, \ldots, n-1, \tag{3}$$

$$a_k^{(2)} = \begin{vmatrix} a_0^{(1)} & a_{n-1-k}^{(1)} \\ a_{n-1}^{(1)} & a_k^{(1)} \end{vmatrix} = a_0^{(1)} a_k^{(1)} - a_{n-1}^{(1)} a_{n-1-k}^{(1)},\ k = 0, 1, \ldots, n-2, \tag{4}$$

$$a_k^{(3)} = \begin{vmatrix} a_0^{(2)} & a_{n-2-k}^{(2)} \\ a_{n-2}^{(2)} & a_k^{(2)} \end{vmatrix} = a_0^{(2)} a_k^{(2)} - a_{n-2}^{(2)} a_{n-2-k}^{(2)},\ k = 0, 1, \ldots, n-3, \tag{5}$$

$$\vdots$$

$$a_k^{(i)} = \begin{vmatrix} a_0^{(i-1)} & a_{n-i+1-k}^{(i-1)} \\ a_{n-i+1}^{(i-1)} & a_k^{(i-1)} \end{vmatrix} = a_0^{(i-1)} a_k^{(i-1)} - a_{n-i+1}^{(i-1)} a_{n-i+1-k}^{(i-1)},\quad k = 0, 1, \ldots, n-i, \tag{6}$$

$$\vdots$$

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$$a_k^{(n-2)} = \begin{vmatrix} a_0^{(n-3)} & a_{3-k}^{(n-3)} \\ a_3^{(n-3)} & a_k^{(n-3)} \end{vmatrix} = a_0^{(n-3)}a_k^{(n-3)} - a_3^{(n-3)}a_{3-k}^{(n-3)}, k = 0, 1, 2, \tag{7}$$

the following stability constraints

$$F(1) > 0 \tag{8}$$
$$(-1)^n F(-1) > 0 \tag{9}$$
$$|a_n| > |a_0| \tag{10}$$
$$|a_0^{(i)}| > |a_{n-i}^{(i)}|, \forall i = 1, 2, \ldots, n-2 \tag{11}$$

are satisfied.

3. Main Results

In this section, by inspecting the original proof which was based on the Jury stability test table, we show that the monotonic condition (2) can be relaxed as follows:

Theorem 1: If one of the following conditions

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_2 \geq a_1 > a_0 > 0 \tag{12}$$
$$a_n > a_{n-1} \geq a_{n-2} \geq \ldots \geq a_2 \geq a_1 \geq a_0 > 0 \tag{13}$$

is satisfied, then $F(z)$ is Schur stable.

Proof: For the case of $n = 1$, it is obvious that the condition $a_1 > a_0 > 0$ implies the Schur stability of the polynomial $F(z) = a_0 + a_1 z$. For $n \geq 2$, we construct the Jury stability test table for the polynomial $F(z)$ as in Lemma 2. From the condition (12) or (13), we see that the test conditions (8), (9) and (10) for the row 0's elements $a_0, a_1, \ldots, a_n$

$$F(1) = \sum_{i=0}^{n} a_i > 0 \tag{14}$$
$$(-1)^n F(-1)_{|n \text{ even}} = a_0 + (a_2 - a_1) + \ldots + (a_n - a_{n-1}) > 0 \tag{15}$$
$$(-1)^n F(-1)_{|n \text{ odd}} = (a_1 - a_0) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) > 0 \tag{16}$$
$$|a_n| > |a_0|, \tag{17}$$

are satisfied.

Now we calculate the row 1's elements $a_0^{(1)}, a_1^{(1)}, \ldots, a_{n-1}^{(1)}$ as in (3) and note that due to the condition (12) or (13), we have

$$a_k^{(1)} - a_{k+1}^{(1)} = a_0(a_k - a_{k+1}) + a_n(a_{n-1-k} - a_{n-k}) \leq 0, k = 0, 1, \ldots, n-2 \tag{18}$$
$$a_0^{(1)} - a_1^{(1)} = a_0(a_0 - a_1) + a_n(a_{n-1} - a_n) < 0 \tag{19}$$
$$a_{n-1}^{(1)} = \begin{vmatrix} a_0 & a_1 \\ a_n & a_{n-1} \end{vmatrix} = a_0 a_{n-1} - a_1 a_n < 0. \tag{20}$$

Therefore, the row 1's elements satisfy

$$0 > a_{n-1}^{(1)} \geq a_{n-2}^{(1)} \geq \ldots \geq a_2^{(1)} \geq a_1^{(1)} > a_0^{(1)}, \tag{21}$$

i.e., the test condition on row 1, $|a_0^{(1)}| > |a_{n-1}^{(1)}|$ is satisfied.

We continue to calculate the row 2's elements $a_0^{(2)}, a_1^{(2)}, \ldots, a_{n-2}^{(2)}$ as in (4) and because of the condition (21), we have

$$a_k^{(2)} - a_{k+1}^{(2)} = a_0^{(1)}(a_k^{(1)} - a_{k+1}^{(1)}) + a_{n-1}^{(1)}(a_{n-2-k}^{(1)} - a_{n-1-k}^{(1)}) \geq 0, k = 0, 1, \ldots, n-3 \tag{22}$$
$$a_0^{(2)} - a_1^{(2)} = a_0^{(1)}(a_0^{(1)} - a_1^{(1)}) + a_{n-1}^{(1)}(a_{n-2}^{(1)} - a_{n-1}^{(1)}) > 0 \tag{23}$$
$$a_{n-2}^{(2)} = \begin{vmatrix} a_0^{(1)} & a_1^{(1)} \\ a_{n-1}^{(1)} & a_{n-2}^{(1)} \end{vmatrix} = a_0^{(1)}a_{n-2}^{(1)} - a_1^{(1)}a_{n-1}^{(1)} > 0. \tag{24}$$

These conditions show that the row 2's elements satisfy

$$a_0^{(2)} > a_1^{(2)} \geq a_2^{(2)} \geq \ldots \geq a_{n-3}^{(2)} \geq a_{n-2}^{(2)} > 0, \tag{25}$$

i.e., the test condition on row 2, $|a_0^{(2)}| > |a_{n-2}^{(2)}|$ is satisfied. Similarly, we calculate the row 3's elements $a_0^{(3)}$, $a_1^{(3)}$, $\ldots$, $a_{n-3}^{(3)}$ as in (5) and have

$$a_0^{(3)} > a_1^{(3)} \geq a_2^{(3)} \geq \ldots \geq a_{n-4}^{(3)} \geq a_{n-3}^{(3)} > 0, \tag{26}$$

i.e., the test condition on row 3, $|a_0^{(3)}| > |a_{n-3}^{(3)}|$ is satisfied.

We note that from row 2 on, the inequality arrangement remains unchanged as in (25) and (26), which are different from that of row 0 and row 1 as in (12), (13) and (21). Hence, the row i's elements $a_0^{(i)}, a_1^{(i)}, \ldots, a_{n-i}^{(i)}$ for $2 \leq i \leq n-2$ calculated as in (6) lead to the following inequality arrangement

$$a_0^{(i)} > a_1^{(i)} \geq a_2^{(i)} \geq \ldots \geq a_{n-i-1}^{(i)} \geq a_{n-i}^{(i)} > 0, \tag{27}$$

that is, the test condition on row i, $|a_0^{(i)}| > |a_{n-i}^{(i)}|$ is satisfied. The final row $(n-2)$'s elements $a_0^{(n-2)}, a_1^{(n-2)}, a_2^{(n-2)}$ have the following relation

$$a_0^{(n-2)} > a_1^{(n-2)} \geq a_2^{(n-2)} > 0, \tag{28}$$

which satisfies the test condition $|a_0^{(n-2)}| > |a_2^{(n-2)}|$. Since the test condition on every row is satisfied, we conclude that $F(z)$ is a Schur stable polynomial.

We see that the condition (12) or (13) requires the strict inequality conditions $a_0 > 0$ and $a_1 > a_0$ or $a_n > a_{n-1}$. Depending on the parity of n, the condition (2) can be relaxed in another way as follows.

Theorem 2: If n is even and one of the following conditions

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_2 > a_1 \geq a_0 > 0 \tag{29}$$
$$a_n \geq a_{n-1} > a_{n-2} \geq \ldots \geq a_2 \geq a_1 \geq a_0 > 0 \tag{30}$$

is satisfied or if n is odd and beside the condition (29) or (30) the following condition

$$(a_n - a_{n-1}) + (a_{n-2} - a_{n-3}) + \ldots + (a_1 - a_0) > 0 \tag{31}$$

is also satisfied, then $F(z)$ is Schur stable.

Proof: The proof of this theorem is briefly outlined as follows. Depending on the parity of n, the condition (29) or (30) for n even and plus the condition (31) for n

odd imply the test conditions on row 0 given in (8), (9) and (10) respectively. We calculate the row 1's elements $a_0^{(1)}, a_1^{(1)}, \ldots, a_{n-1}^{(1)}$ as in (3) and note that due to the condition (29) or (30) we have

$$a_1^{(1)} - a_2^{(1)} = a_0(a_1 - a_2) + a_n(a_{n-2} - a_{n-1}) < 0 \quad (32)$$

$$a_{n-1}^{(1)} = a_0 a_{n-1} - a_1 a_n \leq 0. \quad (33)$$

Therefore the row 1's elements satisfy

$$0 \geq a_{n-1}^{(1)} \geq a_{n-2}^{(1)} \geq \ldots \geq a_2^{(1)} > a_1^{(1)} \geq a_0^{(1)}, \quad (34)$$

i.e., the test condition on row 1, $|a_0^{(1)}| > |a_{n-1}^{(1)}|$ is satisfied.

Similarly, the row i's elements $a_0^{(i)}, a_1^{(i)}, \ldots, a_{n-i}^{(i)}$ for $2 \leq i \leq n-2$ have the following inequality arrangement

$$a_0^{(i)} \geq a_1^{(i)} > a_2^{(i)} \geq \ldots \geq a_{n-i-1}^{(i)} \geq a_{n-i}^{(i)} \geq 0, \quad (35)$$

that is, the test condition on row i, $|a_0^{(i)}| > |a_{n-i}^{(i)}|$ is satisfied. The final row $(n-2)$'s elements $a_0^{(n-2)}, a_1^{(n-2)}, a_2^{(n-2)}$ have the following arrangement

$$a_0^{(n-2)} \geq a_1^{(n-2)} > a_2^{(n-2)} \geq 0, \quad (36)$$

which satisfy the test condition $|a_0^{(n-2)}| > |a_2^{(n-2)}|$. Thus the test condition on every row is satisfied. It is easy to recognize that for odd n we need the extra condition (31) to satisfy (16), while for even n (15) is fulfilled without more constraint. This completes the proof.

Pertaining to the above results, some comments are given:

1) It is possible that if $a_0 = a_1 = \ldots = a_N = 0$, $N \leq n-2$, then the monotonic conditions can be still applied by regarding the first positive coefficient $a_{N+1} > 0$ as a_0 in (1). For the case $N = n-1$ then $F(z) = a_n z^n$ is Schur stable for any $a_n \neq 0$.

2) The polynomials $F^{(i)}(z) = a_{n-i}^{(i)} + a_{n-i-1}^{(i)} z + \ldots + a_1^{(i)} z^{n-i-1} + a_0^{(i)} z^{n-i}$ for $1 \leq i \leq n-2$, where $a_0^{(i)}, a_1^{(i)}, \ldots, a_{n-i}^{(i)}$ are the row i's elements of the Jury test table determined under the conditions of Theorem 1 or 2, are also Schur stable. This can be confirmed by the fact that the conditions of Theorem 1 or 2 for $F^{(i)}(z)$ are successively "inherited" from those of $F^{(i-1)}(z)$ as suggested in the proof of these theorems.

3) The monotonic condition for n odd is stricter than that of n even. The need for the extra condition (31) implies that in this case $z = -1$ could be critical (near) to a real root of $F(z)$.

4) From the above results, one may expect that the stability can be still assured if an inequality is placed in the series between a_{n-2} and a_2 for $n \geq 5$. To show that the expectation is not true in general, we give an example of the polynomial $F(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5$ with $a = a_5 = a_4 = a_3 > a_2 = a_1 = a_0 = b > 0$. Obviously, $F(z) = (b + az^3)(1 + z + z^2)$ is Schur unstable since it has unstable roots $z = (-1 \pm j\sqrt{3})/2$.

5) The obtained results also provide a way to relax the condition given in [2] for a real polynomial to have all its roots outside the unit circle stated as follows. If the conditions in Theorem 1 or 2 are satisfied, then the polynomial $G(z) = a_n + a_{n-1}z + a_{n-2}z^2 + \ldots + a_2 z^{n-2} + a_1 z^{n-1} + a_0 z^n$ has all its roots outside the unit circle.

6) Several conditions for Hurwitz stability of polynomials are derived in [3] by combining the monotonic conditions with bilinear mapping. The results of this note contribute a relaxation of these Hurwitz counterparts as well.

7) Finally we look into the obtained results through the polynomial coefficient space. For simplicity, consider monic polynomials, i.e., $a_n = 1$ and the Schur stability region in n-dimensional coefficient space. The original monotonic condition says that the region bounded by hyperplanes specified by the inequalities resides in the Schur stability region in the "first" orthant. Now, theorems 1 and 2 indicate that only a few hyperplanes can touch the boundaries of the stability region and the rest are detached from them.

4. Conclusion

We have provided several relaxed sufficient monotonic Schur stability conditions for real polynomials. By examining the original proof based on the Jury test method, the original monotonic stability condition with strict inequalities is relaxed in the way that equalities are allowed in all the inequalities but two at appropriate positions. As noted in comment 2, the condition for odd degree polynomials is stricter than that of even degree polynomials. The corresponding Hurwitz stability conditions also can be obtained by using bilinear mapping.

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