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On Large Deviation Theorems for Markov Processes on a Compact Domain

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Abstract

Large deviation theorems of the Donsker-Varadhan type are studied. Those theorems for a family of stochastic processes converging to a Markov process have already been obtained by the present author. In this paper, these theorems are modified so that they cover the case where each process is killed on exiting a compact domain. The general theory of large deviations of such a type is mentioned at two levels; the state-space level and the path-space level, and applied to the study of the principal eigenvalue of the generator of a Markov process. It is shown that the principal eigenvalue converges as the probability law of the corresponding Markov process converges. As a typical example, the converging family of Markov processes in the homogenization problem is investigated.

Key Words: *Large deviation; occupation distribution; Markov process; principal eigenvalue; limit theorem; homogenization.*

1. Introduction

M. D. Donsker and S. R. S. Varadhan established the modern theory of large deviations for a Markov process, and gave outstanding applications.[3, 4, 6, 7, 11] The large deviation theory for a family of stochastic processes converging to a Markov process is established by the present author,[8] where the converging family of stochastic processes can be taken from various limit theorems, for example, the homogenization problem as shown in Example 4.1. On the other hand, the variational formula for the principal eigenvalue of the generator of Markov process was also studied by Donsker-Varadhan.[2, 5] All the results mentioned above are closely related to each other through the variational formula concerned with the "I-function" (see, *e.g.*, eq. (2.8) and eq. (4.3)).

Some results on large deviations for a converging family of Markov processes in the case where the processes are killed on exiting a compact domain, and on the continuity of the principal eigenvalue were found by Ôkura.[9] In this paper, we give full details of these results in Sections 2 and 4, and prove them. In addition to the large deviation theorems at the state-space level given in Section 2, we also give those at the path-space level in Section 3, which are of interest in their own right. Some parts (the lower bound) of Section 2 will be reduced to the general results in Section 3. In Section 4, we apply the general theory in Section 2 to the variational formula for the principal eigenvalue, and establish the continuity of the principal eigenvalue with respect to the probability law of the corresponding Markov process, in the sense of the weak convergence on the path-space. We also give an example of a converging family of Markov processes in a homogenization problem,[1, 10] and show that the corresponding principal eigenvalue also converges.

2. Large Deviation Theorems at the State-Space Level

Let X be a locally compact separable metric space and let Ω^+ be a space of all trajectories on X that are right continuous with no discontinuities of the second kind, endowed with the Skorohod-type topology (see Section 3). For any $B \subset X$, two types of exit times of $\omega \in \Omega^+$ from B are defined by

$$\tau_B(\omega) := \inf\{t \geq 0;\ \omega(t) \notin B\}, \tag{2.1}$$

$$\tau_B^*(\omega) := \inf\{t \geq 0;\ \omega(t) \notin B \text{ or } \omega(t-) \notin B\}, \tag{2.2}$$

where $\omega(t-)$ denotes the left hand limit of ω at $t \geq 0$ with the convention that $\omega(0-) = \omega(0)$.

Let E be any separable metric space. Throughout this paper, $\mathscr{B}(E)$ denotes the Borel σ-algebra on E and $\mathscr{P}(E)$ the space of all probability measures on E endowed with the weak topology. Let $C(E)$ denote the set of all bounded continuous functions on E.

For any $t > 0$ and $\omega \in \Omega^+$, we define $L_{t,\omega} \in \mathscr{P}(X)$ by

$$L_{t,\omega}(B) := \frac{1}{t}\int_0^t 1_B(\omega(s))ds \qquad (B \in \mathscr{B}(X)), \tag{2.3}$$

which is called the *occupation distribution* (at time t) of ω. Here, and in what follows, 1_B denotes the indicator function of set B. Throughout this paper, by a *Markov process* we mean a family $\mathbf{M} = (P_x)$ of probability measures P_x on Ω^+ such that $(\Omega^+, \omega(t), P_x)$ is a canonical realization of a time-homogeneous, conservative Markov process on X. Let G be an open subset of X and let $C_0(G)$ denote the set of all bounded continuous functions on $\overline{G}$ vanishing on the boundary ∂G. Set

$$p_G(t, x, dy) := P_x(\omega;\ \omega(t) \in dy, \tau_{\overline{G}}(\omega) > t) \qquad (t > 0, x \in \overline{G}). \tag{2.4}$$

We impose two assumptions on $\mathbf{M}$ and G:

(**A**) $X \ni x \mapsto P_x \in \mathscr{P}(\Omega^+)$ is continuous.

(**B**) (1) There exist a finite measure $d\alpha(y)$ on $\overline{G}$ with $\alpha(\partial G) = 0$ and a jointly measurable function $p_G(x,y)$ such that, for any $x \in G$, $p_G(1, x, dy) = p_G(x,y)\, d\alpha(y)$ and $p_G(x,y) > 0$ α-a.a. $y \in \overline{G}$;

(2) For any $B \in \mathscr{B}(\overline{G})$, $x \mapsto p_G(1, x, B)$ is continuous on G;

(3) $P_x(\omega;\ \tau_{\overline{G}}(\omega) = \tau_G^*(\omega)) = 1$ for any $x \in \overline{G}$.

Under the condition (**A**), $\mathbf{M}$ generates a Feller semigroup. Its infinitesimal generator is denoted by $(A, \mathscr{D}(A))$. The *I-function* with respect to $\mathbf{M}$ is given by

$$I(\mu):=\sup\{\int_X \frac{-Au}{u}d\mu;\ u\in\mathscr{D}(A), \inf u>0\}\qquad (\mu\in\mathscr{P}(X)). \tag{2.5}$$

It is easy to see that $I(\mu)$ is lower semi-continuous on $\mathscr{P}(X)$.

Suppose that we are given a family of Markov processes $\mathbf{M}^\varepsilon=(P_x^\varepsilon)$ $(\varepsilon>0)$ satisfying

$(\mathbf{A}^\varepsilon)$ For any $x\in X$, $\{P_y^\varepsilon\}$ converges to P_x in $\mathscr{P}(\Omega^+)$ as $\varepsilon\downarrow 0$ and $y\to x$.
It is easy to see that $(\mathbf{A}^\varepsilon)$ implies $(\mathbf{A})$.

Note that if $\overline{G}$ is compact, then $\mathscr{P}(\overline{G})$ is naturally identified with the compact subset $\{\mu\in\mathscr{P}(X);\ \mu(\overline{G})=1\}$ of $\mathscr{P}(X)$.

Theorem 2.1. *Suppose that* $(\mathbf{A}^\varepsilon)$ *is satisfied, and let* $I(\mu)$ *be the I-function with respect to* $\mathbf{M}$.

(1) *For any closed subset* C *of* $\mathscr{P}(X)$, *it holds that*

$$\limsup_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\sup_{x\in G}P_x^\varepsilon(\omega;\ L_{t,\omega}\in C, \tau_{\overline{G}}(\omega)>t)\le -\inf_{\mu\in C\cap\mathscr{P}(\overline{G})}I(\mu). \tag{2.6}$$

(2) *Suppose that* $(\mathbf{B})$ *is satisfied and let* $\mu\in\mathscr{P}(\overline{G})$. *Then, for any neighborhood* N *of* μ *in* $\mathscr{P}(X)$ *and any compact subset* K *of* G, *it holds that*

$$\liminf_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\inf_{x\in K}P_x^\varepsilon(\omega;\ L_{t,\omega}\in N, \tau_G(\omega)>t)\ge -I(\mu). \tag{2.7}$$

By the argument given by Varadhan,[11] we have the following

Corollary 2.1. *Suppose that* $(\mathbf{A}^\varepsilon)$ *and* $(\mathbf{B})$ *are satisfied and that* $\overline{G}$ *is compact. Let* $\Phi(\mu)$ *be a continuous functional on* $\mathscr{P}(\overline{G})$. *Then, it holds that*

$$\lim_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\sup_{x\in G}E_x^\varepsilon[e^{t\Phi(L_{t,\omega})};\tau_G(\omega)>t]=\sup_{\mu\in\mathscr{P}(\overline{G})}[\Phi(\mu)-I(\mu)], \tag{2.8}$$

where E_x^ε *denotes the expectation with respect to* P_x^ε.

Consider a condition weaker than $(\mathbf{A}^\varepsilon)$:

$(\mathbf{A}_0^\varepsilon)$ For any $x\in X$, $\{P_y^\varepsilon\}$ converges to P_x in the sense of finite-dimensional distribution as $\varepsilon\downarrow 0$ and $y\to x$.
To prove Theorem 2.1 (1) we recall an earlier result:

Theorem 2.2 (Theorem 2.3 by Ôkura[8]). *Suppose that* $(\mathbf{A}_0^\varepsilon)$ *is satisfied. For any compact subset* A *of* $\mathscr{P}(X)$, *it holds that*

$$\limsup_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\sup_{x\in X}P_x^\varepsilon(\omega;\ L_{t,\omega}\in A)\le -\inf_{\mu\in A}I(\mu). \tag{2.9}$$

Proof of Theorem 2.1 (1). For any $\omega\in\Omega^+$, $\tau_{\overline{G}}(\omega)>t$ implies that $L_{t,\omega}(\overline{G})=1$ and hence

$L_{t,\omega}\in\mathscr{P}(\overline{G})$. Thus, eq. (2.6) is reduced to eq. (2.9) with $A=C\cap\mathscr{P}(\overline{G})$ since A is compact. □

The second part of Theorem 2.1 is a corollary to a more general result (Theorem 3.1 (2)) given in Section 3. The proof will be given at the end of Section 3.

3. Large Deviation Theorems at the Path-Space Level

Let X be a Polish space with metric d and let $D(I;X)$ denote the space of X-valued right continuous functions on I with no discontinuities of the second kind, endowed with the Skorohod-type topology, where I is a connected subset of $(-\infty,\infty)$; namely, a sequence $\{\omega_n\}$ in $D(I;X)$ is said to converge to a $\omega\in D(I;X)$ if there exists a sequence $\{\lambda_n\}$ of strictly increasing continuous mappings from I onto itself such that

$$\sup_{t\in I\cap[-T,T]}\{d(\omega_n(\lambda_n(t)),\omega(t))+|\lambda_n(t)-t|\}\to 0 \quad\text{as}\quad n\to\infty \tag{3.1}$$

for every finite $T>0$. It is known that $D(I;X)$ with this topology is a Polish space. We define the evaluation mapping x_t on any $D(I;X)$ by $x_t(\omega):=\omega(t)$ for any $t\in I$. In the following, we denote $\Omega:=D((-\infty,\infty);X)$, $\Omega^+:=D([0,\infty);X)$ and $\Omega_T^0:=D([0,T];X)$.

For any measurable space $(E,\mathscr{E})$ and any sub σ-algebra $\mathscr{F}$ of $\mathscr{E}$, we denote by $B(\mathscr{F})$ the set of all bounded $\mathscr{F}$-measurable functions on E, and by $\mathscr{P}(\mathscr{F})$ the set of all probability measures on $(E,\mathscr{F})$. Note that, in case E is a Polish space, we have $\mathscr{P}(\mathscr{B}(E))=\mathscr{P}(E)$ and that $\mathscr{P}(E)$ is itself a Polish space. For any $\mu,\lambda\in\mathscr{P}(\mathscr{F})$, we define the *entropy* of μ with respect to λ by

$$h_{\mathscr{F}}(\lambda;\mu):=\sup_{\Phi\in\mathscr{B}(\mathscr{F})}\left[\int\Phi\,d\mu-\log\int e^{\Phi}d\lambda\right]. \tag{3.2}$$

The following basic fact was proved in Lemma 2.1 by Donsker-Varadhan:[3)]

Proposition 3.1. *For any $\mu,\lambda\in\mathscr{P}(\mathscr{F})$, $h_{\mathscr{F}}(\lambda;\mu)<\infty$ if and only if μ is absolutely continuous with respect to λ and the Radon-Nikodým derivative $f:=d\mu/d\lambda$ is μ-integrable. If this is the case, then we have*

$$h_{\mathscr{F}}(\lambda;\mu)=\int\log f\,d\mu=\int f\log f\,d\lambda. \tag{3.3}$$

Suppose $-\infty\le s\le t\le\infty$. We denote by $\mathscr{F}_t^s$ the σ-algebra over Ω generated by $\{x_\tau;\ \tau\in[s,t]\cap(-\infty,\infty)\}$. It is known that $\mathscr{B}(\Omega)=\mathscr{F}_\infty^{-\infty}$, and that $\mathscr{B}(\Omega^+)$ and $\mathscr{B}(\Omega_T^0)$ can be naturally identified with $\mathscr{F}_\infty^0$ and $\mathscr{F}_T^0$, respectively $(0\le T<\infty)$. Furthermore, the restriction mappings $Q\mapsto Q_{|\mathscr{F}_\infty^0}$ and $Q\mapsto Q_{|\mathscr{F}_T^0}$ define the following natural inclusion relations:

$$\mathscr{P}(\Omega)\hookrightarrow\mathscr{P}(\mathscr{F}_\infty^0)\hookrightarrow\mathscr{P}(\mathscr{F}_T^0) \tag{3.4}$$

so that we can think as

$$\mathscr{P}(\Omega)\subset\mathscr{P}(\Omega^+)\subset\mathscr{P}(\Omega_T^0). \tag{3.5}$$

In the following we repeatedly use these relations without making explicit reference to them. We define the translation operator θ_t of Ω onto itself by $x_\tau \circ \theta_t = x_{\tau+t}$ for any $t \in \mathbb{R}$ and also on Ω^+ for any $t \geq 0$. Let $\mathscr{P}_S(\Omega)$ denote the set of all stationary measures $Q \in \mathscr{P}(\Omega)$, *i.e.*, $Q \circ \theta_t^{-1} = Q$ for all $t \in \mathbb{R}$. Note that $\mathscr{P}_S(\Omega)$ is a closed subspace of $\mathscr{P}(\Omega)$. Let $\mathscr{P}_E(\Omega) := \{Q \in \mathscr{P}_S(\Omega);\ Q \text{ is ergodic}\}$, which is the set of all extremals of the convex set $\mathscr{P}_S(\Omega)$.

Let $\mathbf{M} = (P_x)$ be a Markov process on X satisfying $(\mathbf{A})$. We define the *entropy* $H(t, Q)$ of $Q \in \mathscr{P}_S(\Omega)$ with respect to $\mathbf{M}$ (at time $t > 0$) by

$$H(t, Q) := \int_\Omega h_{\mathscr{F}_t^0}(P_{\omega(0)}; Q_{0,\omega})\, dQ(\omega), \tag{3.6}$$

where $Q_{0,\omega}$ denotes the regular conditional probability distribution of Q given $\mathscr{F}_0^{-\infty}$. In view of Proposition 3.1, if $H(t, Q) < \infty$, then $Q_{0,\omega}$ is absolutely continuous with respect to $P_{\omega(0)}$ for Q-a.a. ω, and we have

$$H(t, Q) = \int_\Omega dQ \int_\Omega \log \frac{dQ_{0,\omega}}{dP_{\omega(0)}}\bigg|_{\mathscr{F}_t^0} dQ_{0,\omega} = \int_\Omega \log \frac{dQ_{0,\omega}}{dP_{\omega(0)}}\bigg|_{\mathscr{F}_t^0} dQ. \tag{3.7}$$

We simply call $H(Q) := H(1, Q)$ the *entropy* of $Q \in \mathscr{P}_S(\Omega)$ with respect to $\mathbf{M}$. It is known[7] that $H(Q)$ is a lower semicontinuous affine functional on $\mathscr{P}_S(\Omega)$ with values in $[0, \infty]$.

Let $t > 0$ and $\omega \in \Omega^+$. For any $B \in \mathscr{B}(\Omega)$ we define

$$R_{t,\omega}(B) := \frac{1}{t} \int_0^t 1_B(\theta_s \tilde{\omega}^t)\, ds, \tag{3.8}$$

where $\tilde{\omega}^t$ denotes the periodic extension of $\{\omega(\tau);\ 0 \leq \tau < t\}$ to $(-\infty, \infty)$.

We define the mapping $q: \mathscr{P}_S(\Omega) \to \mathscr{P}(X)$ by $q[Q](B) := Q(\omega: \omega(0) \in B)\ (B \in \mathscr{B}(X))$, *i.e.*, $q[Q]$ denotes the one-dimensional marginal of Q. It is easy to see that $q[R_{t,\omega}] = L_{t,\omega}$. The following *contraction principle* is established in Theorem 6.1 by Donsker-Varadhan:[7]

$$\inf_{Q \in \mathscr{P}_S(\Omega),\, q[Q] = \mu} H(Q) = I(\mu) \qquad (\mu \in \mathscr{P}(X)). \tag{3.9}$$

Let F be a closed subset of X and let $\Omega_F := D((-\infty, \infty); F)$, which is a closed subset of Ω. Let $\mathscr{P}_S(\Omega_F)$ denote the set of all stationary $Q \in \mathscr{P}(\Omega_F)$, which is identified with a closed subset $\{Q \in \mathscr{P}_S(\Omega);\ Q(\Omega_F) = 1\}$ of $\mathscr{P}_S(\Omega)$.

Lemma 3.1. *For any $Q \in \mathscr{P}_S(\Omega)$, $q[Q](F) = 1$ if and only if $Q(\Omega_F) = 1$.*
Proof. Suppose $q[Q](F) = 1$. Then, since $Q(\omega;\ \omega(0) \in F) = 1$ and since Q is stationary, we have $Q(\omega;\ \omega(t) \in F$ for all rational numbers$) = 1$ and hence $Q(\Omega_F) = 1$. The "if" part is trivial. □

In the rest of this section, let G be as in Section 2 and let $F := \overline{G}$. Consider a family of Markov processes $\mathbf{M}^\varepsilon = (P_x^\varepsilon)$ on $X\ (\varepsilon > 0)$. The following condition will be assumed for the family $\{\mathbf{M}^\varepsilon\}_{\varepsilon > 0}$ only in the first part of the theorem below:

($\mathbf{A}_1^\varepsilon$) For each $\varepsilon>0$ and any compact subset K of X, $\{P_x^\varepsilon;\ x\in K\}$ is tight as a family of measures on $D([0,1];X)$

Theorem 3.1. *Suppose that $F:=\overline{G}$ is compact. Let $H(Q)$ be the entropy with respect to* $\mathbf{M}$.

(1) *Suppose that* ($\mathbf{A}^\varepsilon$) *and* ($\mathbf{A}_1^\varepsilon$) *are satisfied. For any closed subset C of $\mathscr{P}_S(\Omega)$, it holds that*

$$\limsup_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\sup_{x\in G}P_x^\varepsilon(\omega;\ R_{t,\omega}\in C,\tau_F(\omega)>t)\le-\inf_{Q\in C\cap\mathscr{P}_S(\Omega_F)}H(Q). \tag{3.10}$$

(2) *Suppose that* ($\mathbf{A}^\varepsilon$) *and* ($\mathbf{B}$) *are satisfied. Let $Q\in\mathscr{P}_S(\Omega)$ be such that $q[Q](F)=1$. Then, for any neighborhood N of Q in $\mathscr{P}_S(\Omega)$ and any compact subset K of G, it holds that*

$$\liminf_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\inf_{x\in K}P_x^\varepsilon(\omega;R_{t,\omega}\in N,\tau_G(\omega)>t)\ge-H(Q). \tag{3.11}$$

In the proof of this theorem, we need some preparatory results. The following lemma was proved in Lemma 4.4 by Ôkura:[8)]

Lemma 3.2. *Suppose that* ($\mathbf{A}^\varepsilon$) *is satisfied and that there exists a relatively compact open subset G_1 of X. Let $Q\in\mathscr{P}_S(\Omega)$. Then, for any neighborhood N of Q in $\mathscr{P}_S(\Omega)$, there exist an $s_0>0$ and a neighborhood N' of Q in $\mathscr{P}_S(\Omega)$ such that for any open set $G\subset X$, any compact set $K\subset X$ and any $s\ge s_0$, it holds that*

$$\begin{aligned}&\liminf_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\inf_{x\in K}P_x^\varepsilon(\omega;R_{t,\omega}\in N,\tau_G^*(\omega)>t)\\&\ge\frac{1}{s}\log\inf_{x\in K\cup\overline{G_1}}P_x(\omega;R_{s,\omega}\in N',\tau_G^*(\omega)>s,\omega(s)\in G_1).\end{aligned} \tag{3.12}$$

The following is a modification of Theorem 5.5 from Donsker-Varadhan:[7)]

Theorem 3.2. *Suppose that* ($\mathbf{A}$) *and* ($\mathbf{B}$) *are satisfied. Let $Q\in\mathscr{P}_S(\Omega)$ be such that $q[Q](F)=1$. Let G_1 be an open subset of F with $\alpha(G_1)>0$. Then, for any neighborhood N of Q in $\mathscr{P}_S(\Omega)$ and any compact subset K of G, it holds that*

$$\liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in K}P_x(\omega;\ R_{t,\omega}\in N,\tau_F(\omega)>t,\omega(t)\in G_1)\ge-H(Q). \tag{3.13}$$

This may be of interest by itself. The proof will be deferred for a while.

Proof of Theorem 3.1 The first assertion is a corollary to Theorem 2.1 from Ôkura,[8)] which shows that if A is a closed subset of $\mathscr{P}_S(\Omega)$ such that $q[A]$ is a tight family in $\mathscr{P}(X)$, then

$$\limsup_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\sup_{x\in X}P_x^\varepsilon(\omega;\ R_{t,\omega}\in A)\le-\inf_{Q\in A}H(Q). \tag{3.14}$$

Note that, for any $\omega\in\Omega^+$, $\tau_F(\omega)>t$ implies that $R_{t,\omega}(\Omega_F)=1$, namely, $R_{t,\omega}\in\mathscr{P}_S(\Omega_F)$. Thus, eq. (3.10) is reduced to eq. (3.14) with $A := C \cap \mathscr{P}_S(\Omega_F)$ since $q[A]$ is included in compact set $\mathscr{P}(F)$. Turning to the lower bound, we take an open subset G_1 of G such that $\overline{G_1} \subset G$ and $\alpha(G_1) > 0$. Since $\tau_G \geq \tau_G^*$ and $P_x(\tau_F = \tau_G^*) = 1$ $(x \in G)$, eq. (3.11) is reduced to eq. (3.13) by Lemma 3.2. □

In the following, we are concerned with the proof of Theorem 3.2. We need a technical lemma:

Lemma 3.3. *Let $Q \in \mathscr{P}_S(\Omega)$ be such that $H(Q) < \infty$ and $Q(\Omega_F) = 1$. For any $\varepsilon > 0$ and any neighborhood N of Q in $\mathscr{P}(\Omega)$, there exist $Q_k \in \mathscr{P}_E(\Omega)$ with $Q_k(\Omega_F) = 1$ and $c_k > 0$ $(k = 1, 2, \ldots, n)$ such that $\sum_{k=1}^n c_k = 1$, $Q' := \sum_{k=1}^n c_k Q_k \in N$ and $|H(Q) - H(Q')| < \varepsilon$.*

Proof. It is shown in the proof of Lemma 3.4 by Donsker-Varadhan[7] that there exist $\Omega_0 \in \mathscr{F}_0^{-\infty}$ and an $\mathscr{F}_0^{-\infty}$-measurable mapping $\Omega_0 \ni \omega \mapsto \pi_\omega \in \mathscr{P}_E(\Omega)$ such that $Q(\Omega_0) = 1$ for any $Q \in \mathscr{P}_S(\Omega)$, and $Q(\omega; \pi_\omega = Q) = 1$ for any $Q \in \mathscr{P}_E(\Omega)$. Thus, it holds that

$$\int_{\Omega_0} \pi_\omega dQ(\omega) = Q \tag{3.15}$$

for every extremal $Q \in \mathscr{P}_E(\Omega)$. Since this relation is linear in Q, this also holds for any $Q \in \mathscr{P}_S(\Omega)$. It is also shown in the proof of Theorem 3.5 by Donsker-Varadhan[7] that there exists a nonnegative function h_0 on Ω such that $H(Q) = \int h_0(\omega)\, dQ(\omega)$ for any $Q \in \mathscr{P}_S(\Omega)$.

In the following, Q is fixed and suppose that $H(Q) < \infty$ and $Q(\Omega_F) = 1$. Since N is a weak neighborhood of Q, we can find $h_j \in C(\Omega)$ $(j = 1, 2, \ldots, l)$ such that

$$N' := \{Q' \in \mathscr{P}(\Omega);\ |\int h_j dQ' - \int h_j dQ| < 2\varepsilon\ (j = 1, 2, \ldots, l)\} \subset N. \tag{3.16}$$

It follows from eq. (3.15) that $\pi_\omega(\Omega_F) = 1$ Q-a.e. We can take a partition $B_k (k = 1, 2, \ldots, n)$ of Ω_0 such that, for each k, $Q(B_k) > 0$ and

$$\sup_{\omega, \omega' \in B_k} |\int h_j d\pi_\omega - \int h_j d\pi_{\omega'}| < \varepsilon \qquad (j = 0, 1, 2, \ldots, l). \tag{3.17}$$

We can choose $\omega_k \in B_k$ so that $\pi_{\omega_k}(\Omega_F) = 1$. Let $Q' := \sum_{k=1}^n c_k Q_k$ with $c_k := Q(B_k)$ and $Q_k := \pi_{\omega_k}$. It follows from eq. (3.17) that

$$\begin{aligned} |\int h_j dQ - \int h_j dQ'| &= |\int dQ(\omega) \int h_j d\pi_\omega - \sum_{k=1}^n Q(B_k) \int h_j d\pi_{\omega_k}| \\ &\leq \sum_{k=1}^n \int_{B_k} |\int h_j d\pi_\omega - \int h_j d\pi_{\omega_k}| dQ < \varepsilon \end{aligned}$$

$(j = 0, 1, 2, \ldots, l)$. This implies that $|H(Q) - H(Q')| < \varepsilon$ and $Q' \in N' \subset N$. □

By using this lemma and the same argument as in the proof of Theorem 5.5 by Donsker-Varadhan,[7], we can reduce Theorem 3.2 to the following, which is also a modification of Theorem 5.4 by Donsker-Varadhan:[7]

Theorem 3.3. *Suppose that* (**A**) *and* (**B**) *are satisfied. Let* $Q \in \mathscr{P}_E(\Omega)$ *be such that* $H(Q) < \infty$ *and* $q[Q](F) = 1$. *Let* G_1 *be an open subset of* F *with* $\alpha(G_1) > 0$. *Then, for any neighborhood* N *of* Q *in* $\mathscr{P}_S(\Omega)$ *and any compact subset* K *of* G, *it holds that*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in K} P_x(\omega;\ R_{t,\omega} \in N, \tau_F(\omega) > t, \omega(t) \in G_1) \geq -H(Q). \tag{3.18}$$

To prove this we need a technical lemma:

Lemma 3.4. *Suppose that* (**B**) *is satisfied. Let* $Q \in \mathscr{P}_S(\Omega)$ *be such that* $H(Q) < \infty$ *and* $\mu(F) = 1$ *with* $\mu := q[Q]$. *If* $0 \leq \phi_n(y) \leq 1\ (y \in X)$ *and* $\int \phi_n(y) d\mu(y) \geq c_1\ (n = 1, 2, \ldots)$ *for some constant* $c_1 > 0$. *Then, for any compact subset* K *of* G, *there exists a constant* $c_2 > 0$ *such that*

$$\inf_{x\in K} \int \phi_n(y) p_G(1, x, dy) \geq c_2. \tag{3.19}$$

We omit the proof since it is almost the same as that of Lemma 5.3 by Donsker-Varadhan,[7] where the case $G = F = X$ was treated.

Proof of Theorem 3.3. Note that this theorem has been already proved[7] in the case where $G = F = X$. The proof in our case can be done in a similar manner. We only show the difference. Let $\mu := q[Q]$. Take a compact set $K_1 \subset G$ with $\mu(K_1) > 0$. Then there exists a neighborhood N' of Q such that

$$\begin{aligned}
&\inf_{x\in K} P_x(R_{t+2,\omega} \in N, \tau_F(\omega) > t+2, \omega(t+2) \in G_1) \\
&\quad \geq \inf_{x\in K} \int p_G(1, x, dy) P_y(R_{t,\omega} \in N', \tau_F(\omega) > t, \omega(t) \in K_1) \inf_{z\in K_1} p_G(1, z, G_1)
\end{aligned}$$

for sufficiently large t. Let

$$\psi(\omega, t) := \log \frac{dQ_{0,\omega}}{dP_{\omega(0)}}\bigg|_{\mathscr{F}_t^0} \quad \text{and} \quad D_t := \{\omega : R_{t,\omega} \in N', \omega(t) \in K_1\}.$$

Since $Q(\tau_F > t) = 1$ for any $t > 0$ by Lemma 3.1, we have

$$\begin{aligned}
&\int_X d\mu(x) P_x(D_t \cap \{\tau_F(\omega) > t\}) \\
&= \int_\Omega dQ(\omega) P_{\omega(0)}(D_t \cap \{\tau_F(\omega) > t\}) \\
&\geq \int_\Omega dQ(\omega) \int_{D_t \cap \{\tau_F(\omega) > t\}} \left(\frac{dQ_{0,\omega}}{dP_{\omega(0)}}\right)^{-1} dQ_{0,\omega} \\
&= \int_\Omega dQ(\omega) \int_{D_t \cap \{\tau_F(\omega) > t\}} e^{-\psi(\omega,t)} dQ_{0,\omega} \\
&\geq e^{-t(H(Q)+\varepsilon)} Q(\{\tfrac{1}{t}\psi(\omega, t) \leq H(Q) + \epsilon\} \cap D_t \cap \{\tau_F(\omega) > t\}) \\
&\geq e^{-t(H(Q)+\varepsilon)} Q(\{\tfrac{1}{t}\psi(\omega, t) \leq H(Q) + \epsilon\} \cap D_t).
\end{aligned}$$

Since Q is ergodic, the rest is the same as in the proof of Theorem 5.4 by Donsker-Varadhan[7] if we use Lemma 3.4 above instead of Lemma 5.3 by Donsker-Varadhan.[7] □
Thus, we have completed the proof for Theorem 3.2. Finally, we provide the proof for the second half part of the main theorem in Section 2.

Proof of Theorem 2.1 (2). We can assume that $I(\mu) < \infty$ without loss of generality. Since $H(Q)$ is lower semicontinuous, it follows from the contraction principle (3.9) that there exists $Q \in \mathscr{P}_S(\Omega)$ such that $I(\mu) = H(Q)$. Since $q[R_{t,\omega}] = L_{t,\omega}$ and since $q^{-1}[N]$ is a neighborhood of Q in $\mathscr{P}_S(\Omega)$, it follows from eq. (3.11) that

$$\begin{aligned}&\liminf_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\inf_{x\in K}P_x^{\varepsilon}(\omega; L_{t,\omega}\in N, \tau_G(\omega)>t).\\ &=\liminf_{t\to\infty,\varepsilon\downarrow 0}\frac{1}{t}\log\inf_{x\in K}P_x^{\varepsilon}(\omega; R_{t,\omega}\in q^{-1}[N], \tau_G(\omega)>t)\\ &\geq -H(Q) = -I(\mu).\end{aligned}$$ □

4. Continuity of the Principal Eigenvalue

Let X and G be as before and let $V \in C(\overline{G})$. Let $\mathbf{M}$ be a Markov process satisfying (**A**). A semigroup $\{T_t^{V,G}\}$ on $C(\overline{G})$ is defined by

$$T_t^{V,G} f(x) := E_x[f(\omega(t))\, e^{\int_0^t V(\omega(s))ds}; \tau_G(\omega) > t] \qquad (f \in C(\overline{G})). \tag{4.1}$$

Throughout this section we always assume that the following condition holds:
(**T**) Semigroup $\{T_t^{V,G}\}$ is strongly continuous on $C_0(G)$.
Let $A^{V,G}$ be the infinitesimal generator of $\{T_t^{V,G}\}$ and let $\sigma_{V,G}$ denote the spectrum of $A^{V,G}$. Since $t \mapsto \log\sup_{x\in G} T_t^{V,G} 1(x)$ is subadditive, we can always define

$$\lambda_{V,G}[\mathbf{M}] = \lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in G} E_x[e^{\int_0^t V(\omega(s))ds}; \tau_G(\omega) > t]. \tag{4.2}$$

The following result is essentially due to Donsker-Varadhan:[2, 5)]

Proposition 4.1. Suppose that (**A**) and (**T**) are satisfied.
(1) *It holds that* $\lambda_{V,G}[\mathbf{M}] \in \sigma_{V,G}$;
(2) *If* $\mathrm{Re}\, z > \lambda_{V,G}[\mathbf{M}]$ $(z \in \mathbb{C})$, *then* $z \notin \sigma_{V,G}$;
(3) *Moreover, it holds that*

$$\lambda_{V,G}[\mathbf{M}] = \sup_{\mu\in\mathscr{P}(\overline{G})}\Big[\int_{\overline{G}} V(x)d\mu(x) - I(\mu)\Big]. \tag{4.3}$$

We call $\lambda_{V,G}[\mathbf{M}]$ the *principal eigenvalue* of $\mathbf{M}$ for potential V and region G.

Proof of Proposition 4.1. The variational formula (4.3) follows from Corollary 2.1. with

$P_x^\varepsilon = P_x\ (\varepsilon > 0)$ and $\Phi(\mu) = \int_{\overline{G}} V\, d\mu$. The rest is the same as in the proof of Theorem 2.2 by Donsker-Varadhan.[5)] □

Let $\mathfrak{M}$ denote the totality of all Markov processes $\mathbf{M}$ satisfying $(\mathbf{A})$ and $(\mathbf{T})$. Let $C(X;\mathscr{P}(\Omega^+))$ denote the space of all continuous mappings of X into $\mathscr{P}(\Omega^+)$ endowed with the compact-open topology. Since any $\mathbf{M} \in \mathfrak{M}$ satisfies $(\mathbf{A})$, $\mathfrak{M}$ can be considered as a subset of $C(X;\mathscr{P}(\Omega^+))$. Let $\mathfrak{M}_0 := \{\mathbf{M} \in \mathfrak{M};\ \mathbf{M} \text{ satisfies } (\mathbf{B})\}$.

Theorem 4.1. *The mapping* $\mathfrak{M} \ni \mathbf{M} \mapsto \lambda_{V,G}[\mathbf{M}]$ *is continuous on* $\mathfrak{M}_0$.

Proof. Suppose that a family of Markov processes $\mathbf{M}^\varepsilon = (P_x^\varepsilon) \in \mathfrak{M}\ (\varepsilon > 0)$ converges to a Markov process $\mathbf{M} = (P_x) \in \mathfrak{M}_0$. It suffices to show that

$$\lim_{\varepsilon \to 0} \lambda_{V,G}[\mathbf{M}^\varepsilon] = \lambda_{V,G}[\mathbf{M}]. \tag{4.4}$$

This follows from (2.8) with $\Phi(\mu) = \int_{\overline{G}} V\, d\mu$, and (4.2) since $(\mathbf{A}^\varepsilon)$ is satisfied. □

Example 4.1. We recall the result from Osada,[10)] where a homogenization problem for the diffusion processes and operators with random stationary coefficients was investigated. Its general framework also covers the case of periodic or almost periodic coefficients. Let $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ be a probability space and let $\{\hat{T}_x\}\ (x \in \mathbb{R}^d)$ be a d-dimensional stationary ergodic flow on $\hat{\Omega}$. Let $L^2(\hat{\Omega})$ be the real L^2-space on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ and let $\{U_x\}$ denote a strongly continuous unitary group on $L^2(\hat{\Omega})$ defined by $U_x f(\hat{\omega}) := f(\hat{T}_x \hat{\omega})\ (x \in \mathbb{R}^d, \hat{\omega} \in \hat{\Omega}, f \in L^2(\hat{\Omega}))$. For each $i \in \{1, 2, \ldots, d\}$, let D_i denote the infinitesimal generator of $\{U_x\}$ in the i-th direction with domain $\mathscr{D}(D_i)$, namely,

$$D_i f(\hat{\omega}) := \frac{\partial}{\partial x^i} (U_x f)(\hat{\omega})|_{x=0}, \tag{4.5}$$

where the differentiation is taken in the sense of $L^2(\hat{\Omega})$. Let $H^1(\hat{\Omega}) := \bigcap_{i=1}^d \mathscr{D}(D_i)$. For example, the case of periodic coefficients can be realized as follows: Let $\hat{\Omega} := [0,1)^d$, $d\hat{P}(\hat{\omega}) := d\hat{\omega}$ (the Lebesgue measure) and $\hat{T}_x \hat{\omega} := \hat{\omega} + x \pmod 1$. Note that $D_i = \partial/\partial x^i$ in this particular case.

Let $\hat{m}(\hat{\omega})$, $\hat{a}^{ij}(\hat{\omega})$ and $\hat{b}^i(\hat{\omega})\ (i, j = 1, 2, \ldots, d)$ be real-valued measurable functions on $(\hat{\Omega}, \hat{\mathscr{F}})$. Consider a formal differential operator

$$A^{\hat{\omega}} := \frac{1}{m(x,\hat{\omega})} \sum_{i,j=1}^d \frac{\partial}{\partial x^j}\left(a^{ij}(x,\hat{\omega}) \frac{\partial}{\partial x^i} \right) + \sum_{i=1}^d b^i(x,\hat{\omega}) \frac{\partial}{\partial x^i}, \tag{4.6}$$

where $m(x,\hat{\omega}) = \hat{m}(\hat{T}_x \hat{\omega})$, $a^{ij}(x,\hat{\omega}) = \hat{a}^{ij}(\hat{T}_x \hat{\omega})$ and $b^i(x,\hat{\omega}) = \hat{b}^i(\hat{T}_x \hat{\omega})$. Suppose that there exist positive constants k, ν and M, and functions $\hat{c}^{ij} \in H^1(\hat{\Omega})\ (i, j = 1, 2, \ldots, d)$ such that

(1) $1/k \le \hat{m}(\hat{\omega}) \le k$;

(2) $\sum_{i,j=1}^{d} \hat{a}^{ij}(\hat{\omega})\xi^i \xi^j \geq \nu |\xi|^2$ for all $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d$ and $|\hat{a}^{ij}(\hat{\omega})| \leq M$;

(3) $\hat{b}^i(\hat{\omega}) = \sum_{j=1}^{d} D_j \hat{c}^{ij}(\hat{\omega})$ and $|\hat{c}^{ij}(\hat{\omega})| \leq M$;

(4) $\sum_{i=1}^{d} D_i \hat{b}^i = 0$ in the following sense:

$$\int_{\hat{\Omega}} \sum_{i=1}^{d} \hat{b}^i(\hat{\omega}) D_i \phi(\hat{\omega}) \hat{P}(d\hat{\omega}) = 0 \quad \text{for all } \phi \in H^1(\hat{\Omega}). \tag{4.7}$$

Then it is known[10] (see also Example 5.3 and Remark 5.4 by Ôkura[8]) that there exists a set $\hat{\Omega}_0 \subset \hat{\Omega}$ with $\hat{P}(\hat{\Omega}_0) = 1$ such that, for every $\hat{\omega} \in \hat{\Omega}_0$, there exists a unique fundamental solution $p^{\hat{\omega}}(t,x,y)$ for $\partial/\partial t - A^{\hat{\omega}}$. The fundamental solution $p^{\hat{\omega}}(t,x,y)$ is so nice that we can construct a diffusion (*i.e.*, continuous Markov) process $\mathbf{M}^{\hat{\omega}} := (P_x^{\hat{\omega}})$ on $X := \mathbb{R}^d$ having $p^{\hat{\omega}}(t,x,y)$ as its transition density function relative to the Lebesgue measure. For any $\varepsilon > 0$, let $P_x^{\hat{\omega},\varepsilon}$ denote the probability measure on Ω^+ induced from $P_{x/\varepsilon}^{\hat{\omega}}$ by

$$x_t^\varepsilon(\omega) := \varepsilon\omega(t/\varepsilon^2) \qquad (t \geq 0, \omega \in \Omega^+). \tag{4.8}$$

Then for each $\hat{\omega} \in \hat{\Omega}_0$ and $\varepsilon > 0$, $\mathbf{M}^{\hat{\omega},\varepsilon} := (P_x^{\hat{\omega},\varepsilon})$ is the diffusion process associated with

$$A^{\hat{\omega},\varepsilon} := m(\tfrac{x}{\varepsilon}, \hat{\omega})^{-1} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(a^{ij}(\tfrac{x}{\varepsilon}, \hat{\omega}) \frac{\partial}{\partial x^i} \right) + \frac{1}{\varepsilon} \sum_{i=1}^{d} b^i(\tfrac{x}{\varepsilon}, \hat{\omega}) \frac{\partial}{\partial x^i}. \tag{4.9}$$

It is known[10] that $\mathbf{M}^{\hat{\omega}}$ is *homogenizable* in the following sense: There exists a set $\hat{\Omega}_1 \subset \hat{\Omega}_0$ with $\hat{P}(\hat{\Omega}_1) = 1$ such that, for each $\hat{\omega} \in \hat{\Omega}_1$, $\{P_0^{\hat{\omega},\varepsilon}\}$ converges to P_0 in $\mathscr{P}(\Omega^+)$ as $\varepsilon \to 0$, where $\mathbf{M} := (P_x)$ denotes the diffusion process (Brownian motion) generated by

$$A := m^* \sum_{i,j=1}^{d} q^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \qquad \left(m^* := \left(\int_{\hat{\Omega}} \hat{m}\, d\hat{P}\right)^{-1}\right) \tag{4.10}$$

with (q^{ij}) being a certain non-degenerate covariance matrix. Note that $\mathbf{M}$ is independent of $\hat{\omega}$ by the ergodicity of the flow $\{\hat{T}_x\}$. It is known[8] that, for each $\hat{\omega} \in \hat{\Omega}_1$, the family $\{\mathbf{M}^{\hat{\omega},\varepsilon}\}_{\varepsilon>0}$ and $\mathbf{M}$ satisfy conditions $(\mathbf{A}^\varepsilon)$ and $(\mathbf{A}_1^\varepsilon)$. Suppose G is an open subset of X satisfying the *outer-cone condition* in the following sense: For any $x \in \partial G$, there exists a non-empty open cone $C_x := C_x(h, \Theta)$ with vertex x such that $C_x \cap \overline{G} = \emptyset$, where $h > 0$, $\Theta \subset S^{d-1}$, and

$$C_x(h, \Theta) := \{x + r\theta \in \mathbb{R}^d;\ 0 < r < h, \theta \in \Theta\}. \tag{4.11}$$

Since $\mathbf{M}, \mathbf{M}^{\hat{\omega},\varepsilon} \in \mathfrak{M}_0$ $(\hat{\omega} \in \hat{\Omega}_1, \varepsilon > 0)$, we have the following result, which is motivated by Bensoussan *et al.*[1]

Theorem 4.2. Suppose that G is a relatively compact open subset of X satisfying the outer-cone condition in the above sense. Then it holds that, for any $V \in C(\overline{G})$,

$$\lim_{\varepsilon\to 0}\lambda_{V,G}[\mathbf{M}^{\hat{\omega},\varepsilon}]=\lambda_{V,G}[\mathbf{M}] \qquad \textit{for } \hat{P}\textit{-a.a. } \hat{\omega}. \tag{4.12}$$

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