LETTER

Scattering of a TM Plane Wave from a Periodic Surface with Finite Extent: Perturbation Solution

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SUMMARY This paper studies the scattering of a TM plane wave from a perfectly conductive sinusoidal surface with finite extent by the small perturbation method. We obtain the first and second order perturbed solutions explicitly, in terms of which the differential scattering cross section and the total scattering cross section per unit surface are calculated and are illustrated in figures. By comparison with results by a numerical method, it is concluded that the perturbed solution is reasonable even for a critical angle of incidence if the surface is small in roughness and gentle in slope and if the corrugation width is less than certain value. A brief discussion is given on multiple scattering effects.
key words: *scattering of TM wave, periodic surface with finite extent, perturbation method, optical theorem*

1. Introduction

This paper deals with the scattering of a TM plane wave from a periodic surface with finite extent shown in Fig. 1, where W is the width of corrugation. We discuss an analytical solution by the small perturbation method.

Rayleigh [1] and Rice [2] studied the diffraction by a periodic surface with infinite extent ($W = \infty$) by the small perturbation method. They found the perturbation solution diverges unphysically when the angle of incidence becomes critical and one of the Floquet modes is diffracted into a grazing direction.

However, this paper points out that, when W is finite, the perturbation solution has several desirable properties. First, the perturbation solution is free from such a divergence problem for any angle of incidence. Second, it satisfies the reciprocity and the optical theorem up to the second order with respect to the surface roughness. Third, it remains finite for a non-critical angle of incidence even when $W \to \infty$. Fourth, it is reasonable even for a critical angle of incidence, if W is not very wide. When the angle of incidence is critical and when $W \to \infty$, however, the perturbation solution gives unphysical divergence of the total scattering cross section per unit surface.

Then, we discuss the validity of the perturbation solution in terms of W. We find that the perturbed solution is reasonable even for a critical angle of incidence if the surface is small in roughness and gentle in slope and if the corrugation width is less than certain value.

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2. Formulation

Let us consider the scattering of a TM plane wave from the sinusoidal surface with finite extent shown in Fig. 1. We write the surface corrugation as

$$z = f(x) = \sigma u(x|W)\sin(k_L x), \quad k_L = \frac{2\pi}{L}, \tag{1}$$

where σ is the surface roughness, k_L is the spatial angular frequency of the period L, W is the width of corrugation and $u(x|W)$ is a rectangular pulse,

$$u(x|W) = u^2(x|W) = \begin{cases} 1, & |x| \le W/2 \\ 0, & |x| > W/2 \end{cases}. \tag{2}$$

We denote the y component of the magnetic field by $\psi(x,z)$, which satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\psi(x,z) = 0, \tag{3}$$

in the region $z > f(x)$ and the Neumann condition on the surface (1)

$$\left[\frac{\partial}{\partial z} - \frac{df}{dx}\frac{\partial}{\partial x}\right]\psi(x,z)\bigg|_{z=f(x)} = 0. \tag{4}$$

Here, $k = 2\pi/\lambda$ is wave number and λ is wavelength. We write the incident plane wave $\psi_i(x,z)$ as

$$\psi_i(x,z) = e^{-ipx}e^{-i\beta(p)z}, \quad p = k \cdot \cos\theta_i, \tag{5}$$

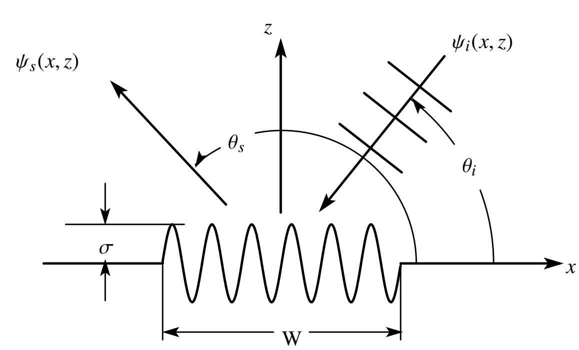

Fig. 1 The scattering of a TM plane wave from a periodic surface with finite extent. $\psi_i(x,z)$ and $\psi_s(x,z)$ are the incident plane wave and the scattered wave, respectively. θ_i and θ_s are the angle of incidence and a scattering angle, respectively, which are measured from the positive x axis. W and σ are the width and roughness of the surface, respectively.

where θ_i is the angle of incidence measured from the positive x axis as shown in Fig. 1, and $\beta(p)$ is a function of p given by

$$\beta(p) = \sqrt{k^2 - p^2},$$
$$Re\,[\beta(p)] \geq 0, \quad Im\,[\beta(p)] \geq 0. \tag{6}$$

Here, Re and Im are real and imaginary parts. Since the surface is flat when $|x| > W/2$, we put

$$\psi(x,z) = \psi_i(x,z) + e^{-ipx}e^{i\beta(p)z} + \psi_s(x,z), \tag{7}$$

where the second term on the right-hand side is the specularly reflected wave and $\psi_s(x,z)$ is the scattered wave due to the surface roughness. In far region, $\psi_s(x,z)$ becomes a cylindrical wave and hence its Fourier spectrum $A(s)$ has singularities. But its angular spectrum $A_\beta(s) = \beta(p+s)A(s)$ is always finite, as is discussed in Ref. [3]. Using this fact and assuming the Rayleigh hypothesis, we write $\psi_s(x,z)$ as

$$\psi_s(x,z) = \int_{-\infty}^{\infty} \frac{A_\beta(s)}{\beta(p+s)} e^{-i(p+s)x + i\beta(p+s)z} ds. \tag{8}$$

Here, the angular spectrum $A_\beta(s)$ gives the amplitude of the partial wave scattered into $\theta_s = \Theta(p+s)$ direction, where $\Theta(p+s)$ is defined by

$$\Theta(p+s) = \arccos\left[-\frac{p+s}{k}\right] = \int_{-(p+k)}^{s} \frac{ds'}{\beta(p+s')}, \tag{9}$$

which is a key equation of this paper. This relation is easily obtained if we change the variable of integration from s' to α by the relation $p+s' = -k\cos(\alpha)$. If we put $s = mk_L$, $(m = 0, \pm1, \pm2, \cdots)$, (9) becomes the famous grating formula [4] for a perfectly periodic surface,

$$\Theta(p + mk_L) = \arccos[-(p + mk_L)/k], \tag{10}$$

where $\Theta(p + mk_L)$ is the mth order diffraction angle. Physically, the scattered wave is a sum of diffraction beams, a main lobe of which is scattered into $\theta_s = \Theta(p + mk_L)$ direction.

The optical theorem is analogous to the famous forward scattering theorem. It may be written as [3], [5]

$$p_c = p_{inc}, \tag{11}$$
$$p_c = -\frac{4\pi}{k} Re[A_\beta(0)], \tag{12}$$
$$p_{inc} = \frac{2\pi}{k} \int_{-\infty}^{\infty} Re[\beta_n(p+s)] \left|\frac{A_\beta(s)}{\beta(p+s)}\right|^2 ds \tag{13}$$
$$= \frac{W}{2\pi} \int_0^{\pi} \sigma_s(\theta_s|\theta_i) d\theta_s, \tag{14}$$
$$\sigma_s(\theta_s|\theta_i) = \frac{4\pi^2}{kW} |A_\beta(-k\cos\theta_s - k\cos\theta_i)|^2. \tag{15}$$

The optical theorem (11) states that the total scattering cross section p_{inc} is equal to p_c the loss of the amplitude of the partial wave scattered into the specularly reflected direction. Because of (11), however, we will call p_c the total scattering cross section. Here, $\sigma_s(\theta_s|\theta_i)$ is the differential scattering cross section per unit surface. However, it is important to note that the optical theorem could be a bridge between the scattering from a finite periodic surface and the diffraction by a periodic surface with infinite extent [6], [7].

3. Small Perturbation

Let us determine the angular spectrum by the small perturbation method. Assuming $k\sigma \ll 1$, we expand $A_\beta(s)$ into a power series of σ as

$$A_\beta(s) = \sigma A_\beta^{(1)}(s) + \sigma^2 A_\beta^{(2)}(s) + \cdots. \tag{16}$$

After some manipulation, we obtain the 1st and 2nd order perturbed solution as

$$A_\beta^{(1)}(s) = \frac{1}{2\pi}[k^2 - p(p+s)]U^{(d)}(s|W), \tag{17}$$
$$A_\beta^{(2)}(s) = -\int_{-\infty}^{\infty} U^{(d)}(s'|W)U^{(d)}(s'-s|W) \times \frac{[k^2 - p(p+s')][k^2 - (p+s')(p+s)]}{8\pi^2\beta(p+s')} ds', \tag{18}$$
$$U^{(d)}(s|W) = U(s - k_L|W) - U(s + k_L|W), \tag{19}$$

where $U(s|W)$ is the Fourier transform of $u(x|W)$,

$$U(s|W) = \int_{-\infty}^{\infty} u(x|W)e^{isx}dx = W\frac{\sin\left(\frac{Ws}{2}\right)}{\left(\frac{Ws}{2}\right)}. \tag{20}$$

Physically, $A_\beta^{(1)}(s)$ and $A_\beta^{(2)}(s)$ represent the single scattering and bare double scattering, respectively.

3.1 Properties of Perturbed Solution

Our perturbation solution has several desirable properties. First, we consider the reciprocity. Substituting (17), (18) and (16) into (15), and putting $p = k\cos(\theta_i)$, one easily finds that our perturbation solution satisfies the reciprocity $\sigma_s(\theta_s|\theta_i) = \sigma_s(\theta_i|\theta_s)$.

Next, let us see some numerical examples. For numerical calculations, we put

$$L = 2.5\lambda, \quad \sigma = 0.05\lambda, \tag{21}$$
$$W = 50\lambda, \quad \theta_i = 53.130°. \tag{22}$$

Since $L = 2.5\lambda$, the critical angle of incidence becomes $\theta_i = 0°$, 53.130° and 78.463°. Anomalies at $\theta_i = 0°$ and 78.463° are weak, because they are generated by the 2nd order perturbation. Putting $\theta_i = 53.130°$, we mainly consider effects of the 1st order perturbation below.

Using (17) and (18), we calculated $\sigma_s(\theta_s|\theta_i)$ in Fig. 2(A), where the 1st and −1st order diffraction beams by the 1st order perturbation appear at $\theta_s = 180°$ and 101.53°, respectively, and the 0th and −2nd order diffraction beams by the 2nd order perturbation appear at $\theta_s = 126.87°$ and 78.463°. Note that the 1st order diffraction beam at $\theta_s = 180°$ has much wide beam width. On the other hand,

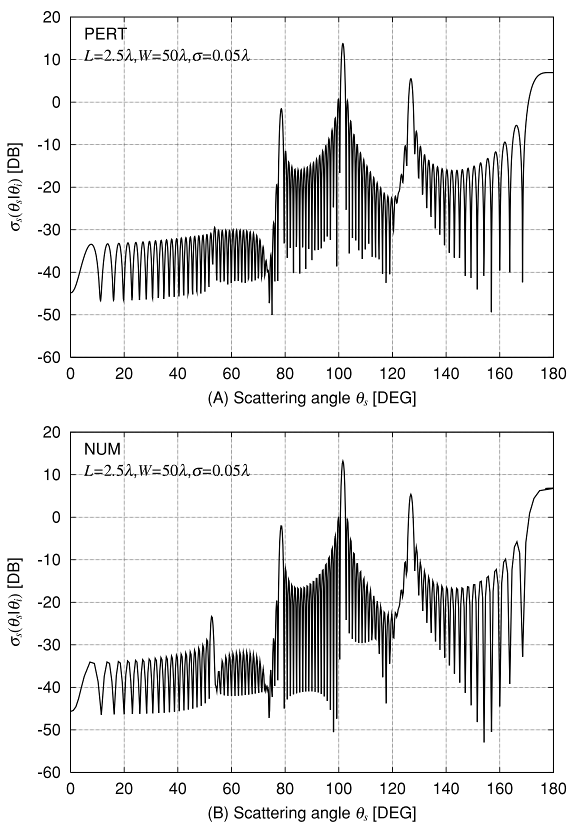

Fig. 2 Scattering cross section. (A) Perturbation, (B) numerical solution. $\theta_i = 53.130°$, $L = 2.5\lambda$, $W = 50\lambda$, $\sigma = 0.05\lambda$.

Fig. 2(B) is obtained by an accurate numerical method in Ref. [3]. We see good agreement between these figures, except for the −3rd order diffraction peak at $\theta_s = 53.130°$, which is small in level and may be obtained by the 3rd order perturbation. Thus we may conclude that, when θ_i is critical, our perturbation solution becomes valid if the width W is not very wide. This point will be discussed later.

Let us consider the optical theorem. Since $A_\beta^{(1)}(0) = 0$ by (17), from (17) and (18) we obtain

$$p_c^{(2)} = -\frac{4\pi\sigma^2}{k} Re[A_\beta^{(2)}(0)] = \frac{2\pi\sigma^2}{k}\int_{-\infty}^{\infty} Re[\beta(p+s')]\left|\frac{A_\beta^{(1)}(s)}{\beta(p+s)}\right|^2 ds, \quad (23)$$

which is a σ^2 order quantity. This means that our perturbation solution satisfies the optical theorem up to the σ^2 order.

Let us evaluate the integral in (23). When $W \gg L$, we roughly regard $U(s|W)$ as a rectangular pulse $Wu(s|k_W)$, where $k_W = 2\pi/W$. Then, we obtain from (17) and (23) a rough estimation,

$$\frac{p_c^{(2)}}{W\sigma^2} \approx W\frac{[k^2 - p(p+k_L)]^2}{2\pi k} Re[\Delta\Theta(p+k_L)] + W\frac{[k^2 - p(p-k_L)]^2}{2\pi k} Re[\Delta\Theta(p-k_L)], \quad (24)$$

where we have used the relation (9). Here, $\Delta\Theta(p+s)$ is the beam width given by

$$\Delta\Theta(p+s) = \Theta(p+s+k_W/2) - \Theta(p+s-k_W/2) \approx \begin{cases} k_W/\beta(p+s), & p+s \neq \pm k \\ (1-i)\sqrt{k_W/k}, & p+s = \pm k \end{cases}, \quad (25)$$

where $k_W = 2\pi/W$. Equations (24) and (25) are the main result of this paper. If θ_i is not critical and $p \pm k_L \neq \pm k$, the beam width $\Delta\Theta(p \pm k_L)$ is proportional to λ/W by (25). Therefore, the total scattering cross section per unit surface $p_c^{(2)}/W$ becomes independent of W and proportional σ^2 by (24). However, when $\theta_i = 53.130°$ and $p + k_L = k$, for example, we obtain $\Delta\Theta(p+k_L) = \Delta\Theta(k) \sim \sqrt{\lambda/W}$ by (25). As a result, the 1st order diffraction beam scattered into $\theta_s = 180°$ has a much wide beam width as is shown in Fig. 2. Furthermore, $p_c^{(2)}/W$ is proportional to $\sigma^2\sqrt{W/\lambda}$, which is a remarkable result of this paper. This means that $p_c^{(2)}/W$ diverges unphysically when θ_i is critical and when $W \to \infty$. Such unphysical divergence is the same drawback as in the Rayleigh-Rice perturbation theory of diffraction grating [1], [2].

4. Validity and Conclusion

When $W \to \infty$ and the surface becomes perfectly periodic, we have the Floquet solution, which may be written as [8]

$$\psi(x,z) = \psi_i(x,z) + e^{-ipx}e^{i\beta(p)z} + \sum_{m=-\infty}^{\infty} A_m e^{-i(p+mk_L)x}e^{i\beta(p+mk_L)z}, \quad (26)$$

where A_m, $(m \neq 0)$, is the amplitude of the mth order Floquet mode and $(A_0 + 1)$ is the reflection coefficient.

Physically, the scattered wave (7) from a finite periodic surface is expected to converge to the diffracted wave (26) by the perfectly periodic surface when $W \to \infty$. Such convergence is doubtful in mathematical sense. But we may expect the total scattering cross section per unit surface p_c/W converges to $p_c^{(g)}$ the total diffraction cross section per unit surface of the periodic case,

$$\lim_{W\to\infty}\frac{p_c}{W} = p_c^{(g)}. \quad (27)$$

Here, $p_c^{(g)}$ satisfies $0 \leq p_c^{(g)} \leq 4\sin(\theta_i)$ and is given by the amplitude A_0 as [8],

$$p_c^{(g)} = -2\frac{\beta(p)}{k}Re[A_0]. \quad (28)$$

Figure 3 shows $p_c^{(2)}/W$ by (21) and (23) for $\theta_i = 30°$ and 53.130°. When $\theta_i = 30°$ and $W > 10^2\lambda$, $p_c^{(2)}/W$ is almost constant equal to 3.963×10^{-2}, whereas (24) gives $p_c^{(2)}/W = 3.966 \times 10^{-2}$. These values are much close to $p_c^{(g)} = 3.908 \times 10^{-2}$ calculated by a rigorous grating theory [9]. This means that the perturbation solution is reasonable for a non-critical angle of incidence if the surface roughness σ is small enough. This also suggests that the optical theorem could be a bridge between the scattering from

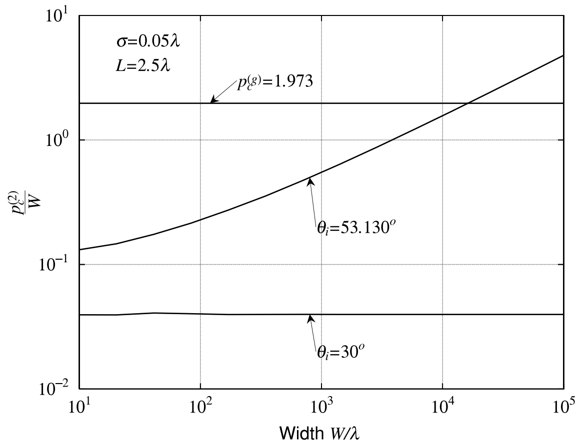

Fig. 3 Total scattering cross section per unit surface by small perturbation method. $p_c^{(g)} = 1.973$ is the total diffraction cross section per unit surface at $\theta_i = 53.130°$. $L = 2.5\lambda$, $\sigma = 0.05\lambda$.

a finite periodic surface and the diffraction by a perfectly periodic surface [6], [7]. On the other hand, for a critical angle of incidence $\theta_i = 53.130°$, the grating theory gives $p_c^{(g)} = 1.973$ but this figure shows $p_c^{(2)}/W \approx 0.01516\sqrt{W/\lambda}$ when $W > 10^4\lambda$, whereas $p_c^{(2)}/W \approx 0.01579\sqrt{W/\lambda}$ by (24). Since the curve $p_c^{(2)}/W$ intersects the line $p_c^{(g)} = 1.973$ at $W = W_i \approx 1.7 \times 10^4\lambda$, we may expect our perturbation solution gives a reasonable result for W much less than $W_i \approx 1.7 \times 10^4\lambda$. Examples of $p_c^{(2)}/W$ by the perturbation (p_c/W by the numerical method) are 0.186 (0.187) at $W/\lambda = 50$, 0.290 (0.308) at $W/\lambda = 200$, and 0.377 (0.400) at $W/\lambda = 400$. As W/λ increases, discrepancy between $p_c^{(2)}/W$ and p_c/W becomes large but is still less than 6% at $W/\lambda = 400$.

Since $0 \leq p_c^{(g)} \leq 4\sin(\theta_i)$, we roughly put $p_c^{(g)} = 4\sin(\theta_i)$ to estimate the intersection W_i from (24) and (25). Then, the range W of the validity is roughly estimated as

$$W \ll W_i = \left(\frac{L}{\pi\sigma}\right)^4 \sin^2(\theta_i), \qquad (29)$$

which depends on the surface slope σ/L. Note that (29) holds only for θ_i satisfying $k\cos(\theta_i) \pm k_L = \pm k$.

Let us consider physical processes of the scattering. When θ_i is critical, there exists a diffraction beam scattered into a grazing direction. Such a beam is scattered by the surface corrugation and then re-scattered again. Therefore, the dressed multiple scattering [10] may take place when W is large. Because our perturbation solution does not take such multiple scattering effects into consideration, it unphysical diverges when $W \to \infty$. When W is much less than W_i, however, such the multiple scattering is weak and the scattered wave is mainly generated by the single and bare double scattering processes. Thus, we may conclude that our perturbation solution gives a reasonable result if the surface is small in roughness and gentle in slope and if the width of corrugation W is much less than W_i.

Our discussions were limited to the 1st and 2nd order perturbed solutions. In order to obtain a reasonable solution for $W \gg W_i$, a new multiple scattering theory has to be developed. Several numerical methods exist for analysis [3], [11], [12] but they are all restricted for a case with W/λ less than about 10^2. To analyze a case with $W/\lambda \gg 10^4$, a new numerical method must be developed. These problems are, however, left for future study.

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