PAPER

Periodic Fourier Transform and Its Application to Wave Scattering from a Finite Periodic Surface: Two-Dimensional Case

Junichi NAKAYAMA[†a)], ***Member***

SUMMARY In this paper, the previously introduced periodic Fourier transform concept is extended to a two-dimensional case. The relations between the periodic Fourier transform, harmonic series representation and Fourier integral representation are also discussed. As a simple application of the periodic Fourier transform, the scattering of a scalar wave from a finite periodic surface with weight is studied. It is shown that the scattered wave may have an extended Floquet form, which is physically considered as the sum of diffraction beams. By the small perturbation method, the first order solution is given explicitly and the scattering cross section is calculated.

key words: *bigrating, diffraction beam, periodic Fourier transform, harmonic series representation, s-periodic spectrum, harmonic spectrum*

1. Introduction

The wave scattering from a periodic surface with finite extent has received much interest, because any real periodic structure is finite in extent and finite periodic structures have important applications. Several methods for analysis were previously introduced [1]–[6]. In a previous study [7], however, we proposed the periodic Fourier transform as a new tool for analysis. The periodic Fourier transform converts any function into a periodic spectrum function with a parameter s. The inverse transform is given by a Fourier integral with s over a finite interval. Considering the periodic Fourier transform of the scattered wave and expanding the periodic spectrum function into a Fourier series, it is shown that the scattered wave has an extended Floquet form, which is considered as the sum of diffraction beams. Using the periodic Fourier transform, we presented a new formulation for the wave scattering from a finite corrugated plane [8], [9] and an apodised periodic surface [10].

However, our discussions were limited to a one-dimensional case. In this paper, a two-dimensional case is discussed to deal with the wave scattering from a bigrating with finite extent or a bigrating with weight (see Fig. 1). We give the basic definition of the two-dimensional periodic Fourier transform and its relation to the harmonic series representation and Fourier integral expression. As a simple application, we discuss the scattering of a scalar wave from a finite periodic surface with the Gaussian weight. Then, we give the first order perturbed solution, in terms of which the scattering cross section is calculated.

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[†]The author is with the Faculty of Engineering and Design, Kyoto Institute of Technology, Kyoto-shi, 606-8585 Japan.

a) E-mail: nakayama@dj.kit.ac.jp

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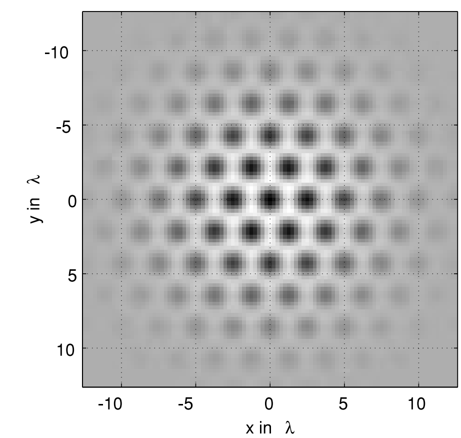

Fig. 1 Two-dimensional periodic structure with Gaussian weight. See Sect. 5.2 for details.

2. Translation Operator and Basic Vectors

Let $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y$ be a two-dimensional vector in the two-dimensional plane $R^2 = (-\infty, \infty)^2$, where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors in the x and y directions, respectively. For a function $f(\mathbf{r})$, we define the translation operator $D^{(n_1,n_2)}$ by

$$D^{(n_1,n_2)} f(\mathbf{r}) = f(\mathbf{r} + \mathbf{L}_1 n_1 + \mathbf{L}_2 n_2), \tag{1}$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are basic vectors, and n_1 and n_2 are any integers. From (1), $D^{(n_1,n_2)}$ becomes a group:

$$D^{(0,0)} = 1, \quad [D^{(n_1,n_2)}]^{-1} = D^{(-n_1,-n_2)},$$
$$D^{(n_1,n_2)} D^{(m_1,m_2)} = D^{(n_1+m_1,n_2+m_2)}, \tag{2}$$

where m_1 and m_2 are any integers also. In general, the basic vectors are not orthogonal to each other and form a parallelogram. Such a parallelogram is called the $\mathbf{r}$ unit cell and is denoted by C_L when its center is located at $\mathbf{r} = (x, y) = 0$ as shown in Fig. 2. We denote the area of the $\mathbf{r}$ unit cell by A_L, which is given by $|\mathbf{L}_1 \times \mathbf{L}_2|$ and is assumed to be positive. We write

$$\mathbf{L}_1 = L_{1x}\mathbf{e}_x + L_{1y}\mathbf{e}_y, \quad \mathbf{L}_2 = L_{2x}\mathbf{e}_x + L_{2y}\mathbf{e}_y, \tag{3}$$

$$A_L = \left\| \begin{matrix} L_{1x}, & L_{2x} \\ L_{1y}, & L_{2y} \end{matrix} \right\| = L_{1x}L_{2y} - L_{2x}L_{1y} \ > \ 0, \tag{4}$$

where $\| \cdot \|$ denotes the determinant.

Let us denote the Bragg basic vectors by $\mathbf{q}_1$ and $\mathbf{q}_2$, which satisfy the orthogonality relation[†]

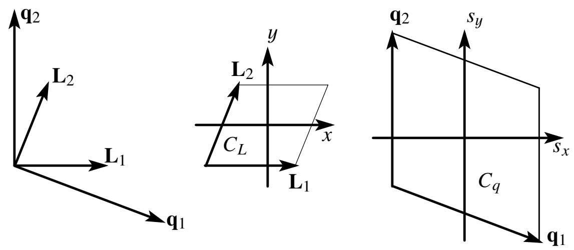

Fig. 2 $\mathbf{L}_1$ and $\mathbf{L}_2$ are basic vectors. $\mathbf{q}_1$ and $\mathbf{q}_2$ are Bragg basic vectors with $\mathbf{L}_i \cdot \mathbf{q}_j = 2\pi\delta_{i,j}$. C_L and C_q are the $\mathbf{r}$ unit cell and the $\mathbf{s}$ unit cell, respectively.

$$\mathbf{L}_i \cdot \mathbf{q}_j = 2\pi\delta_{i,j}, \quad (i, j = 1, 2), \tag{5}$$

where $\delta_{i,j}$ stands for Kronecker's delta. The Bragg basic vectors may be given by

$$\mathbf{q}_1 = \frac{2\pi}{A_L}(L_{2y}\mathbf{e}_x - L_{2x}\mathbf{e}_y),$$
$$\mathbf{q}_2 = \frac{2\pi}{A_L}(L_{1x}\mathbf{e}_y - L_{1y}\mathbf{e}_x). \tag{6}$$

The Bragg basic vectors also form a parallelogram. We call such a parallelogram the **s** unit cell and denote it by C_q when its center is located at $s_x = s_y = 0$ as shown in Fig. 2. We denote the area of the **s** unit cell by A_q,

$$A_q = |\mathbf{q}_1 \times \mathbf{q}_2| = \frac{(2\pi)^2}{A_L}. \tag{7}$$

To simplify notations, we introduce a lattice vector $\mathbf{L}(n_1, n_2)$ and a Bragg lattice vector $\mathbf{q}(m_1, m_2)$ by

$$\mathbf{L}(n_1, n_2) = n_1\mathbf{L}_1 + n_2\mathbf{L}_2, \tag{8}$$
$$\mathbf{q}(m_1, m_2) = m_1\mathbf{q}_1 + m_2\mathbf{q}_2, \tag{9}$$

where n_i and m_j, $(i, j = 1, 2)$, are any integers. In terms of the basic vectors and the Bragg basic vectors, we write

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y = \alpha_1\mathbf{L}_1 + \alpha_2\mathbf{L}_2, \tag{10}$$
$$\mathbf{s} = s_x\mathbf{e}_x + s_y\mathbf{e}_y = \mu_1\mathbf{q}_1 + \mu_2\mathbf{q}_2, \tag{11}$$
$$\mu_1 = \frac{1}{2\pi}\mathbf{s} \cdot \mathbf{L}_1, \quad \mu_2 = \frac{1}{2\pi}\mathbf{s} \cdot \mathbf{L}_2. \tag{12}$$

From (12), the area elements $d\mathbf{r}$ and $d\mathbf{s}$ are given by

$$d\mathbf{r} = dxdy = A_L d\alpha_1 d\alpha_2, \quad d\mathbf{s} = A_q d\mu_1 d\mu_2. \tag{13}$$

By the orthogonality relation (5), one easily obtains

$$\mathbf{s} \cdot \mathbf{r} = xs_x + ys_y = 2\pi(\alpha_1\mu_1 + \alpha_2\mu_2), \tag{14}$$
$$\mathbf{q}(m_1, m_2) \cdot \mathbf{r} = 2\pi(\alpha_1 m_1 + \alpha_2 m_2), \tag{15}$$
$$\mathbf{s} \cdot \mathbf{L}(n_1, n_2) = 2\pi(\mu_1 n_1 + \mu_2 n_2). \tag{16}$$

3. Fourier Series

The Fourier series expression of a multi-dimensional periodic function is well known and is widely applied to solid-state physics [11] and the theory of grating [12]. However, we briefly discuss the Fourier series expression of a two-dimensional periodic function here.

Let $f_p(\mathbf{r})$ be a periodic function with the periods $\mathbf{L}_1$ and $\mathbf{L}_2$, satisfying

$$D^{(n_1,n_2)}f_p(\mathbf{r}) = f_p(\mathbf{r} + \mathbf{L}(n_1, n_2)) = f_p(\mathbf{r}),$$
$$(n_1, n_2 = 0, \pm 1, \pm 2, \cdots). \tag{17}$$

If we write

$$f_\alpha(\alpha_1, \alpha_2) = f_p(\alpha_1\mathbf{L}_1 + \alpha_2\mathbf{L}_2) = f_p(\mathbf{r}), \tag{18}$$

$f_\alpha(\alpha_1, \alpha_2)$ becomes a periodic function with the period 1 in the α_1 and α_2 directions,

$$f_\alpha(\alpha_1, \alpha_2) = f_\alpha(\alpha_1 + n_1, \alpha_2 + n_2), \tag{19}$$

where n_1 and n_2 are any integers. Then, $f_\alpha(\alpha_1, \alpha_2)$ is easily expanded into a Fourier series as

$$f_\alpha(\alpha_1, \alpha_2) = \sum_{m_1,m_2=-\infty}^{\infty} F_{m_1,m_2} \times e^{2\pi i(m_1\alpha_1 + m_2\alpha_2)}, \tag{20}$$

where we assume the uniform convergence of the right-hand side. To calculate the Fourier coefficient F_{m_1,m_2}, we consider the orthogonality relation over C_L,

$$\delta_{m_1,m_1'}\delta_{m_2,m_2'} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-2\pi i[(m_1-m_1')\alpha_1+(m_2-m_2')\alpha_2]}d\alpha_1 d\alpha_2 \tag{21}$$
$$= \frac{1}{A_L}\int_{C_L} e^{-i[\mathbf{q}(m_1,m_2)-\mathbf{q}(m_1',m_2')]\cdot\mathbf{r}}d\mathbf{r}, \tag{22}$$

where we have changed the variables of integration from (α_1, α_2) to $\mathbf{r}$ and used (13) to obtain (22) from (21). Then, the Fourier coefficient F_{m_1,m_2} is calculated as

$$F_{m_1,m_2} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} f_\alpha(\alpha_1, \alpha_2) \times e^{-2\pi i(m_1\alpha_1+m_2\alpha_2)}d\alpha_1 d\alpha_2 \tag{23}$$
$$= \frac{1}{A_L}\int_{C_L} f_p(\mathbf{r})e^{-i\mathbf{q}(m_1,m_2)\cdot\mathbf{r}}d\mathbf{r}. \tag{24}$$

By (15), (20) may be rewritten as

$$f_p(\mathbf{r}) = \sum_{m_1,m_2=-\infty}^{\infty} F_{m_1,m_2}e^{i\mathbf{q}(m_1,m_2)\cdot\mathbf{r}}, \tag{25}$$

which is a well-known Fourier series expression. We note that for a real function $f_p(\mathbf{r})$, the Fourier coefficients satisfy

$$F^*_{m_1,m_2} = F_{-m_1,-m_2}, \tag{26}$$

where the asterisk denotes the complex conjugate.

† In Ref. [11], $\mathbf{q}_1$ and $\mathbf{q}_2$ are defined by $\mathbf{L}_i \cdot \mathbf{q}_j = \delta_{i,j}$ and are called the reciprocal basic vectors. However, we define the Bragg basic vectors by (5) for convenience.

4. Periodic Fourier Transform

The periodic Fourier transform [7] for a one-dimensional function is extended to a two-dimensional one in this section. However, we note that the periodic Fourier transform is considered as a simplified version of the D^a-Fourier transform for an imhomogenous random function [13]. In this section, the harmonic series representation and its relation to the Fourier spectrum (see Fig. 3) are briefly discussed. Several properties of the periodic Fourier transform are listed in Appendix B.

Using the translation operator $D^{(n_1,n_2)}$, we define the periodic Fourier transform of a function $f(\mathbf{r})$ by

$$
\begin{aligned}
F(\mathbf{r},\mathbf{s}) &= \sum_{n_1,n_2=-\infty}^{\infty} D^{(n_1,n_2)}\left[f(\mathbf{r})e^{-i\mathbf{s}\cdot\mathbf{r}}\right] \\
&= \sum_{n_1,n_2=-\infty}^{\infty} f\left(\mathbf{r}+\mathbf{L}(n_1,n_2)\right)e^{-i\mathbf{s}\cdot(\mathbf{r}+\mathbf{L}(n_1,n_2))}, \qquad (27)
\end{aligned}
$$

where we also assume the uniform convergence of the right-hand side. From (27), $F(\mathbf{r},\mathbf{s})$ becomes invariant under $D^{(n_1,n_2)}$ and is a periodic function of $\mathbf{r}$:

$$
F(\mathbf{r},\mathbf{s}) = F(\mathbf{r}+\mathbf{L}(n_1,n_2),\mathbf{s}), \quad (n_1,n_2 = 0,\pm1,\pm2,\cdots). \qquad (28)
$$

Thus, we call $F(\mathbf{r},\mathbf{s})$ the s-periodic spectrum. The s-periodic spectrum is not periodic on $\mathbf{s}$. However, for any integers m_1 and m_2, it satisfies

$$
F(\mathbf{r},\mathbf{s}+\mathbf{q}(m_1,m_2)) = e^{-i\mathbf{q}(m_1,m_2)\cdot\mathbf{r}}F(\mathbf{r},\mathbf{s}). \qquad (29)
$$

To obtain an inversion formula, we modify the periodic Fourier transform (27) using (14) and (16) as

$$
\begin{aligned}
&F(\alpha_1\mathbf{L}_1+\alpha_2\mathbf{L}_2,\mu_1\mathbf{q}_1+\mu_2\mathbf{q}_2)e^{2\pi i(\alpha_1\mu_1+\alpha_2\mu_2)} \\
&\quad= \sum_{n_1,n_2=-\infty}^{\infty} f(\mathbf{r}+\mathbf{L}(n_1,n_2))e^{-2\pi i(\mu_1 n_1+\mu_2 n_2)}. \qquad (30)
\end{aligned}
$$

The right-hand side of this equation is a periodic function of μ_1 and μ_2 and should be understood as a Fourier series, where $f(\mathbf{r}+\mathbf{L}(n_1,n_2))$ is a Fourier coefficient. Since

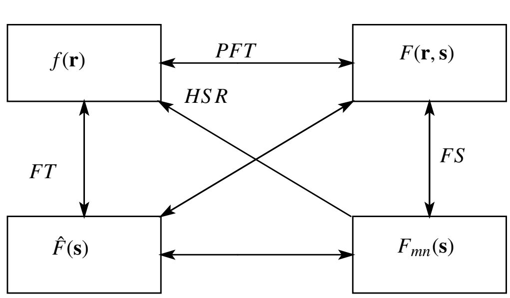

Fig. 3 Periodic Fourier transform (PFT), harmonic series representation (HSR) and Fourier transform (FT). $\hat{F}(s)$ is the Fourier spectrum, $F(\mathbf{r},\mathbf{s})$ is the s-periodic spectrum, and $F_{mn}(s)$ is the harmonic spectrum.

$$
\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-2\pi i(n_1\mu_1+n_2\mu_2)}d\mu_1 d\mu_2 = \delta_{n_1,0}\delta_{n_2,0},
$$

one may easily obtain the inversion formula as

$$
\begin{aligned}
f(\mathbf{r}) &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{2\pi i(\alpha_1\mu_1+\alpha_2\mu_2)} \\
&\quad\times F(\alpha_1\mathbf{L}_1+\alpha_2\mathbf{L}_2,\mu_1\mathbf{q}_1+\mu_2\mathbf{q}_2)d\mu_1 d\mu_2 \qquad (31) \\
&= \frac{1}{A_q}\int_{C_q} F(\mathbf{r},\mathbf{s})e^{i\mathbf{s}\cdot\mathbf{r}}d\mathbf{s}. \qquad (32)
\end{aligned}
$$

4.1 Harmonic Series Representation

Since the spectrum $F(\mathbf{r},\mathbf{s})$ is a periodic function of $\mathbf{r}$, we may obtain another expression for $f(\mathbf{r})$. Using (25), we represent the s-periodic spectrum $F(\mathbf{r},\mathbf{s})$ by a Fourier series,

$$
F(\mathbf{r},\mathbf{s}) = \sum_{m_1,m_2} F_{m_1,m_2}(\mathbf{s})e^{i\mathbf{q}(m_1,m_2)\cdot\mathbf{r}}, \qquad (33)
$$

where $F_{m_1,m_2}(\mathbf{s})$ is calculated by (24). Substituting (33) into (31), we obtain another series representation of $f(\mathbf{r})$,

$$
\begin{aligned}
f(\mathbf{r}) &= f(\alpha_1\mathbf{L}_1+\alpha_2\mathbf{L}_2) \\
&= \sum_{m_1,m_2} e^{2\pi i(m_1\alpha_1+m_2\alpha_2)}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \\
&\quad F_{m_1,m_2}(\mu_1\mathbf{q}_1+\mu_2\mathbf{q}_2)e^{2\pi i(\alpha_1\mu_1+\alpha_2\mu_2)}d\mu_1 d\mu_2 \qquad (34) \\
&= \frac{1}{A_q}\sum_{m_1,m_2} e^{i\mathbf{q}(m_1,m_2)\cdot\mathbf{r}}\int_{C_q} F_{m_1,m_2}(\mathbf{s})e^{i\mathbf{s}\cdot\mathbf{r}}d\mathbf{s}. \qquad (35)
\end{aligned}
$$

We call (35) the harmonic series representation and $F_{m_1,m_2}(\mathbf{s})$ the harmonic spectrum.

4.2 Fourier Spectrum

Let $\hat{F}(\mathbf{s})$ be the Fourier spectrum of $f(\mathbf{r})$. Using the rectangular function $u(\mathbf{s}|C_q)$ defined by (A· 1) and (A· 3), we obtain

$$
\begin{aligned}
f(\mathbf{r}) &= \left(\frac{1}{2\pi}\right)^2\int_{R^2} e^{i\mathbf{s}\cdot\mathbf{r}}\hat{F}(\mathbf{s})d\mathbf{s} \\
&= \sum_{m_1,m_2=-\infty}^{\infty}\int_{R^2} e^{i\mathbf{s}\cdot\mathbf{r}}u(\mathbf{s}-\mathbf{q}(m_1,m_2)|C_q)\frac{\hat{F}(\mathbf{s})}{4\pi^2}d\mathbf{s} \\
&= \left(\frac{1}{2\pi}\right)^2\sum_{m_1,m_2=-\infty}^{\infty} e^{i\mathbf{q}(m_1,m_2)\cdot\mathbf{r}} \\
&\quad\times\int_{C_q} e^{i\mathbf{s}\cdot\mathbf{r}}\hat{F}(\mathbf{s}+\mathbf{q}(m_1,m_2))d\mathbf{s}. \qquad (36)
\end{aligned}
$$

Comparing (36) with (32), we formally obtain the relation between the s-periodic spectrum $F(\mathbf{r},\mathbf{s})$ and the Fourier spectrum $\hat{F}(\mathbf{s})$ as

$$
\begin{aligned}
F(\mathbf{r},\mathbf{s}) &= \frac{1}{A_L}\sum_{m_1,m_2=-\infty}^{\infty} e^{i\mathbf{q}(m_1,m_2)\cdot\mathbf{r}} \\
&\quad\times\hat{F}(\mathbf{s}+\mathbf{q}(m_1,m_2)). \qquad (37)
\end{aligned}
$$

If we compare (37) with (33), we obtain the relation between the Fourier spectrum and the harmonic spectrum $F_{m_1,m_2}(\mathbf{s})$ as

$$F_{m_1,m_2}(\mathbf{s}) = \frac{1}{A_L}\hat{F}(\mathbf{s}+\mathbf{q}(m_1,m_2)). \tag{38}$$

It is important to note that (37) and (38) hold only for $\mathbf{s}$ in C_q. For any $\mathbf{s}$ in R^2, we obtain

$$F_{m_1,m_2}(\mathbf{s})u(\mathbf{s}|C_q) = \frac{1}{A_L}\hat{F}(\mathbf{s}+\mathbf{q}(m_1,m_2))u(\mathbf{s}|C_q).$$

Replacing $\mathbf{s}$ by $\mathbf{s}-\mathbf{q}(m_1,m_2)$ and using (A· 3), we may determine the relation between the Fourier spectrum and the harmonic spectrum as

$$\frac{\hat{F}(\mathbf{s})}{A_L} = \sum_{m_1,m_2=-\infty}^{\infty} F_{m_1,m_2}(\mathbf{s}-\mathbf{q}(m_1,m_2)) \times u(\mathbf{s}-\mathbf{q}(m_1,m_2)|C_q). \tag{39}$$

5. Application to Wave Scattering from Periodic Surface with Finite Extent

As a simple application of the periodic Fourier transform and the harmonic series representation, let us consider the scattering of a scalar wave from a periodically deformed planer surface. We write the periodic deformation with finite extent as

$$z = \sigma_h g(\mathbf{r}) f_p(\mathbf{r}). \tag{40}$$

Here, σ_h is the surface height parameter, $f_p(\mathbf{r})$ is a periodic function with (17) and $g(\mathbf{r})$ is the envelope of the surface deformation with a maximum value at $\mathbf{r} = 0$,

$$\max\{|g(\mathbf{r})|\} = g(0) = 1,$$
$$g(\mathbf{r}) = 0, \quad |\mathbf{r}| > r_{max}, \tag{41}$$
$$g(\mathbf{r}) = \frac{1}{4\pi^2}\int_{R^2} e^{i\mathbf{s}\cdot\mathbf{r}}\hat{G}(\mathbf{s})d\mathbf{s},$$
$$\hat{G}(\mathbf{s}) = \hat{G}^*(-\mathbf{s}), \tag{42}$$
$$A_e = \frac{1}{g(0)}\int_{R^2} g(\mathbf{r})d\mathbf{r} = \frac{\hat{G}(0)}{g(0)}, \tag{43}$$

where r_{max} is a finite number, A_e is the effective area and the asterisk denotes the complex conjugate. By (41) the surface becomes flat when $|\mathbf{r}| > r_{max}$.

We denote the scalar wave field by $\psi(\mathbf{r}, z)$, which satisfies the Helmholtz wave equation

$$[\nabla^2 + k^2]\psi(\mathbf{r}, z) = 0, \tag{44}$$

in free space above the surface and the Dirichlet boundary condition on the surface (40)

$$\psi(\mathbf{r}, z) = 0, \quad (z = \sigma_h g(\mathbf{r}) f_p(\mathbf{r})). \tag{45}$$

Here, $k = 2\pi/\lambda$ is wave number, λ is wavelength, and $\nabla = \mathbf{e}_x\partial/\partial x + \mathbf{e}_y\partial/\partial y + \mathbf{e}_z\partial/\partial z$.

Let us represent the incident plane wave $\psi_i(\mathbf{r}, z)$ and the reflected wave $\psi_r(\mathbf{r}, z)$ by

$$\psi_i(\mathbf{r}, z) = e^{i\mathbf{p}\cdot\mathbf{r} - i\beta_{00}(\mathbf{p})z}, \tag{46}$$
$$\psi_r(\mathbf{r}, z) = -e^{i\mathbf{p}\cdot\mathbf{r} + i\beta_{00}(\mathbf{p})z}, \tag{47}$$

where $\mathbf{p}$ and $-\beta_{00}(\mathbf{p})$ are the projections of the incident wave vector $\mathbf{K}_i$ on the $x-y$ plane and the z axis, respectively (see Fig. 4).

$$\mathbf{K}_i = \mathbf{p} - \beta_{00}(\mathbf{p})\mathbf{e}_z,$$
$$\mathbf{p} = -k(\sin\theta_i\cos\phi_i\mathbf{e}_x + \sin\theta_i\sin\phi_i\mathbf{e}_y). \tag{48}$$

We define $\beta_{mn}(\mathbf{p})$ as a function of $\mathbf{p}$,

$$\beta_{mn}(\mathbf{p}) = \beta_{00}(\mathbf{p} + m\mathbf{q}_1 + n\mathbf{q}_2) = \sqrt{k^2 - (\mathbf{p} + m\mathbf{q}_1 + n\mathbf{q}_2)^2},$$
$$Re[\beta_{mn}(\mathbf{p})] \geq 0, \quad Im[\beta_{mn}(\mathbf{p})] \geq 0, \quad (m, n = 0, \pm1, \pm2, \cdots), \tag{49}$$
$$\beta_{00}(\mathbf{p}) = k\cos\theta_i. \tag{50}$$

Here, Re and Im stand for the real and imaginary parts, respectively, and (θ_i, ϕ_i) is the angle of incidence (see Fig. 4).

We write the total field as the sum of three components: the incident plane wave $\psi_i(\mathbf{r}, z)$, the reflected wave $\psi_r(\mathbf{r}, z)$ and the scattered wave $\psi_s(\mathbf{r}, z)$ due to the surface deformation

$$\psi(\mathbf{r}, z) = \psi_i(\mathbf{r}, z) + \psi_r(\mathbf{r}, z) + \psi_s(\mathbf{r}, z). \tag{51}$$

Assuming the Rayleigh hypothesis, we write the scattered wave as

$$\psi_s(\mathbf{r}, z) = \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{(2\pi)^2}\int_{R^2} e^{i\mathbf{s}\cdot\mathbf{r} + i\beta_{00}(\mathbf{p}+\mathbf{s})z}A(\mathbf{s})d\mathbf{s} \tag{52}$$
$$= \sum_{m,n=-\infty}^{\infty} \frac{e^{i(\mathbf{p}+\mathbf{q}(m,n))\cdot\mathbf{r}}}{A_q} \times \int_{C_q} e^{i\mathbf{s}\cdot\mathbf{r} + i\beta_{mn}(\mathbf{p}+\mathbf{s})z}A_{mn}(\mathbf{s})d\mathbf{s}, \tag{53}$$

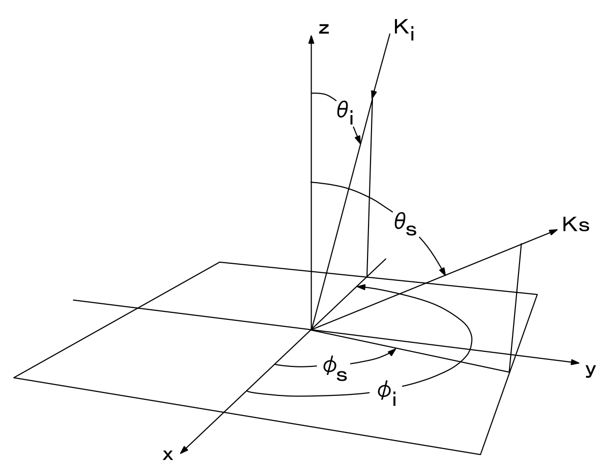

Fig. 4 Scattering of a plane wave. $\mathbf{K}_i$ is the incident wave vector and $\mathbf{K}_s$ is a scattering wave vector.

where (52) is a representation by the Fourier integral. Applying the harmonic series representation to (52) and using (38), we obtain (53). Physically, (53) is given by the sum of $\psi_{mn}(\mathbf{r}, z)$,

$$\psi_{mn}(\mathbf{r}, z) = \frac{e^{i[\mathbf{p}+\mathbf{q}(m,n)]\cdot\mathbf{r}}}{A_q} \times \int_{C_q} e^{i\mathbf{s}\cdot\mathbf{r}+i\beta_{mn}(\mathbf{p}+\mathbf{s})z} A_{mn}(\mathbf{s}) d\mathbf{s}, \tag{54}$$

which we call the (m, n)-th order diffraction beam. The $A_{mn}(\mathbf{s})$ is the amplitude of a plane wave scattered into the $\mathbf{K}_s$ direction,

$$\mathbf{K}_s = \mathbf{p} + \mathbf{s} + \mathbf{q}(m, n) + \beta_{mn}(\mathbf{p} + \mathbf{s})\mathbf{e}_z,$$

$$\mathbf{p} + \mathbf{s} + \mathbf{q}(m, n) = k(\sin\theta_s \cos\phi_s \mathbf{e}_x + \sin\theta_s \sin\phi_s \mathbf{e}_y), \tag{55}$$

$$\beta_{mn}(\mathbf{p} + \mathbf{s}) = k \cos\theta_s, \tag{56}$$

where (θ_s, ϕ_s) is a scattering angle (see Fig. 4). If we put $\mathbf{s} = 0$, (55) is reduced to the grating formula:

$$\mathbf{p} + \mathbf{q}(m, n) = k(\sin\theta_{mn} \cos\phi_{mn} \mathbf{e}_x + \sin\theta_{mn} \sin\phi_{mn} \mathbf{e}_y), \tag{57}$$

where (m, n) stands for the order of diffraction. The diffraction beams of different orders are orthogonal in the sense that

$$Re\left[\frac{1}{ik}\int_{R^2} \psi_{mn}(\mathbf{r}, z) \frac{\partial \psi^*_{m'n'}(\mathbf{r}, z)}{\partial z} d\mathbf{r}\right] = \delta_{mm'}\delta_{nn'}\Phi_{mn}, \tag{58}$$

where $\delta_{mm'}$ is Kronecker's delta and Φ_{mn} is the energy carried by the (m, n)-th order diffraction beam,

$$\Phi_{mn} = \frac{A_L}{kA_q}\int_{C_q} Re[\beta_{mn}(\mathbf{p} + \mathbf{s})]|A_{mn}(\mathbf{s})|^2 d\mathbf{s}. \tag{59}$$

Since the scattering takes place from the corrugated part of the surface, the scattered energy always remains finite. Such a finite energy of the scattering is described by the optical theorem stating that the total scattered energy P_t is proportional to P_c, where P_c is the loss of the amplitude of the plane wave scattered into the specular direction. In our case, the optical theorem may be written as

$$P_c = P_t, \tag{60}$$

where the total scattered energy P_t and the effect of the loss of the specularly scattered amplitude are given by

$$P_c = \frac{2\beta_{00}(\mathbf{p})A_L}{k} Re[A_{00}(0)], \tag{61}$$

$$P_t = \sum_{mn} \Phi_{mn}. \tag{62}$$

Here, Φ_{mn} is given by (59). We denote the differential scattering cross section per unit area by $\sigma(\theta_s, \phi_s|\theta_i, \phi_i)$,

$$P_t = \frac{A_e}{4\pi}\int_0^{\pi/2}\int_0^{2\pi} \sigma(\theta_s, \phi_s|\theta_i, \phi_i) \sin\theta_s d\phi_s d\theta_s, \tag{63}$$

where $\sin\theta_s d\phi_s d\theta_s$ is a differential solid angle. Then we have

$$\sigma(\theta_s, \phi_s|\theta_i, \phi_i) = \frac{A_L^2 k^2 \cos^2\theta_s}{A_e \pi} \sum_{m,n} |A_{mn}(\mathbf{\Lambda} - \mathbf{q}(m, n))|^2 u(\mathbf{\Lambda} - \mathbf{q}(m, n)|C_q), \tag{64}$$

$$\mathbf{\Lambda} = k \sin\theta_s \cos\phi_s \mathbf{e}_x + k \sin\theta_s \sin\phi_s \mathbf{e}_y - \mathbf{p}. \tag{65}$$

Note that the right-hand-side of (64) is divided by the effective area A_e to obtain the scattering cross section per unit area.

Let us consider the relation of (53) to the Floquet form in the grating theory. If we put

$$A_{mn}(\mathbf{s}) = A_q \hat{A}_{mn}\delta(\mathbf{s}), \tag{66}$$

(53) is reduced to the well-known Floquet form for the bigrating [12]

$$\psi_s(\mathbf{r}, z) = \sum_{m,n} \hat{A}_{mn} e^{i(\mathbf{p}+\mathbf{q}(m,n))\cdot\mathbf{r}+i\beta_{mn}(\mathbf{p})z}. \tag{67}$$

Therefore, (53) is considered as an extension of the Floquet form and we call it the extended Floquet form.

5.1 Approximate Solution by Small Perturbation

From (53) and (45), we obtain the integral equation that determines $A_{mn}(\mathbf{s})$,

$$-2i \sin[\sigma_h g(\mathbf{r}) f_p(\mathbf{r})\beta_{00}(\mathbf{p})] + \sum_{m,n=-\infty}^{\infty} \frac{e^{i\mathbf{q}(m,n)\cdot\mathbf{r}}}{A_q} \times \int_{C_q} e^{i\mathbf{s}\cdot\mathbf{r}} e^{i\sigma_h g(\mathbf{r}) f_p(\mathbf{r})\beta_{mn}(\mathbf{p}+\mathbf{s})} A_{mn}(\mathbf{s}) d\mathbf{s} = 0. \tag{68}$$

Assuming a sufficiently small surface deformation, we expand $A_{mn}(\mathbf{s})$ into a perturbation series with respect to σ_h,

$$A_{mn}(\mathbf{s}) = \sigma_h A^{(1)}_{mn}(\mathbf{s}) + \sigma_h^2 A^{(2)}_{mn}(\mathbf{s}) + \cdots. \tag{69}$$

Then we obtain equations for $A^{(1)}_{mn}(\mathbf{s})$ and $A^{(2)}_{mn}(\mathbf{s})$ as

$$-2ig(\mathbf{r}) f_p(\mathbf{r})\beta_{00}(\mathbf{p}) + \sum_{m,n=-\infty}^{\infty} \frac{e^{i\mathbf{q}(m,n)\cdot\mathbf{r}}}{A_q} \times \int_{C_q} e^{i\mathbf{s}\cdot\mathbf{r}} A^{(1)}_{mn}(\mathbf{s}) d\mathbf{s} = 0, \tag{70}$$

$$\sum_{m,n=-\infty}^{\infty} e^{i\mathbf{q}(m,n)\cdot\mathbf{r}}\left[\int_{C_q} e^{i\mathbf{s}\cdot\mathbf{r}} A^{(2)}_{mn}(\mathbf{s}) d\mathbf{s} + ig(\mathbf{r}) f_p(\mathbf{r}) \times \int_{C_q} e^{i\mathbf{s}\cdot\mathbf{r}} \beta_{mn}(\mathbf{p} + \mathbf{s}) A^{(1)}_{mn}(\mathbf{s}) d\mathbf{s}\right] = 0. \tag{71}$$

Using (A· 6) and (A· 8), we calculate the periodic Fourier transform of (70) to obtain

$$-2i\beta_{00}(\mathbf{p}) F_g(\mathbf{r}, \mathbf{s}) f_p(\mathbf{r}) + \sum_{m,n=-\infty}^{\infty} e^{i\mathbf{q}(m,n)\cdot\mathbf{r}} A^{(1)}_{mn}(\mathbf{s}) = 0, \tag{72}$$

which involves only periodic functions of $\mathbf{r}$. Here, $F_g(\mathbf{r}, \mathbf{s})$ is the periodic Fourier transform of $g(\mathbf{r})$,

$$F_g(\mathbf{r}, \mathbf{s}) = \frac{1}{A_L} \sum_{m,n} e^{i\mathbf{q}(m,n)\cdot\mathbf{r}} \hat{G}(\mathbf{s} + \mathbf{q}(m, n)). \tag{73}$$

Using (25) and (73), we easily obtain the first order solution from (72),

$$A^{(1)}_{mn}(\mathbf{s}) = \frac{2i}{A_L}\beta_{00}(\mathbf{p}) \times \sum_{m_1,n_1} \hat{G}(\mathbf{s} + \mathbf{q}(m_1, n_1))F_{m-m_1,n-n_1}, \tag{74}$$

which holds for any $\mathbf{s}$ in the $\mathbf{s}$ unit cell.

Let us assume that $g(\mathbf{r})$ is non-negative and is a simple pulse without ripples. When such a pulse is nearly isotropic and A_e is much wider than A_L, $\hat{G}(\mathbf{s})$ is well localized at $|\mathbf{s}| \approx 0$ and $\hat{G}(\mathbf{s} + \mathbf{q}(m, n)) \approx 0$ holds except for $m = n = 0$. In such a case, we approximately determine

$$A^{(1)}_{mn}(\mathbf{s}) \approx \frac{2i}{A_L}\beta_{00}(\mathbf{p})F_{mn}\hat{G}(\mathbf{s}), \tag{75}$$

which holds for any $\mathbf{s}$ in the $\mathbf{s}$ unit cell. This means that $A^{(1)}_{mn}(\mathbf{s})$ is also localized at $|\mathbf{s}| \approx 0$. Thus, the (m, n)-th order diffraction beam $\psi_{mn}(\mathbf{r}, z)$ is mainly scattered into the (θ_{mn}, ϕ_{mn}) direction given by the grating formula (57). The relation (75) suggests that the diffraction beam shape is much affected by $\hat{G}(\mathbf{s})$.

Since the surface deformation (40) is real and Fourier coefficients satisfy (26), one may obtain

$$Re[A^{(1)}_{00}(0)] = 0. \tag{76}$$

Using (71) and (74), we may obtain the second order solution as

$$\begin{aligned} A^{(2)}_{mn}(\mathbf{s}) &= -i \sum_{m_1,n_1} \sum_{m_3,n_3} F_{m-m_1-m_3,n-n_1-n_3} \int_{C_q} \\ &\frac{\beta_{m_1n_1}(\mathbf{p} + \mathbf{s}')}{A_L A_q} A_{m_1n_1}(\mathbf{s}')\hat{G}(\mathbf{s} - \mathbf{s}' + \mathbf{q}(m_3, n_3))d\mathbf{s}' \\ &= \frac{2\beta_{00}(\mathbf{p})}{A_q A_L^2} \sum_{m_1,n_1} \sum_{m_2,n_2} \sum_{m_3,n_3} F_{m-m_1-m_3,n-n_1-n_3} \\ &\times F_{m_1-m_2,n_1-n_2} \int_{C_q} \beta_{m_1n_1}(\mathbf{p} + \mathbf{s}')\hat{G}(\mathbf{s}' + \mathbf{q}(m_2, n_2)) \\ &\times \hat{G}(\mathbf{s} - \mathbf{s}' + \mathbf{q}(m_3, n_3))d\mathbf{s}'. \end{aligned} \tag{77}$$

Putting $m = n = 0$ and $\mathbf{s} = 0$ in (77), and using (74), we obtain

$$\begin{aligned} &\frac{2\beta_{00}(\mathbf{p})A_L}{k} Re[A^{(2)}_{00}(0)] \\ &= \frac{A_L}{kA_q} \sum_{m,n} \int_{C_q} Re[\beta_{mn}(\mathbf{p} + \mathbf{s})]|A^{(1)}_{mn}(\mathbf{s})|^2 d\mathbf{s}. \end{aligned} \tag{78}$$

By (78) and (76), the optical theorem holds in the order of σ_h^2. From discussions on the applicability of the perturbation method [14], the first order solution (74) is expected to be useful when $\sigma_h \leq 0.1\lambda$.

5.2 Numerical Example

Using the first order solution, we calculate the scattering cross section. In this calculation, we put the basic vectors

$$\begin{aligned} &L_{1x} = L, \quad L_{1y} = 0, \\ &L_{2x} = L\cos(\pi/3), \quad L_{2y} = L\sin(\pi/3). \end{aligned} \tag{79}$$

We set the Fourier coefficients in (25) as

$$\begin{aligned} &F_{1,0} = F_{-1,0} = F_{0,1} = F_{0.-1} = F_{1,1} = F_{-1,-1} = \frac{1}{6}, \\ &F_{m,n} = 0, \quad \text{any other } (m, n), \end{aligned} \tag{80}$$

where only 6 components have non-zero values equal to $1/6$. For simplicity, we consider the Gaussian weight $g(\mathbf{r})$,

$$g(\mathbf{r}) = e^{-\mathbf{r}^2/2\kappa^2}, \quad \hat{G}(\mathbf{s}) = 2\pi\kappa^2 e^{-\kappa^2\mathbf{s}^2/2}, \tag{81}$$

$$A_e = 2\pi\kappa^2, \tag{82}$$

where A_e is the effective area. The Gaussian weight does not satisfy (41) mathematically, but it satisfies such a relation physically. We also put

$$\sigma_h = 0.1\lambda, \quad L = 2.5\lambda, \quad \kappa = 5\lambda, \tag{83}$$

where λ is wavelength. By (79), (80) and (81), the surface deformation (40) is revealed to have a weighted triangular structure that is illustrated as a gray scale image in Fig. 1.

Using the first order solution (74), we numerically calculated the scattering cross section $\sigma(\theta_s, \phi_s|\theta_i, \phi_i)$ as a function of (θ_s, ϕ_s), where we put

$$\theta_i = \pi/6, \quad \phi_i = \pi. \tag{84}$$

The result is shown in Fig. 5, where a black footprint represents a main-lobe of a diffraction beam. Because of the Gaussian weight $g(\mathbf{r})$, there are no ripples and sidelobes associated with a main-lobe. There are only 6 footprints because $f_p(\mathbf{r})$ has 6 non-zero Fourier components given by (80). Since the surface is symmetrical with respect to the x axis and $\phi_i = \pi$, the footprints are symmetrical with respect to the line $\phi_s = \pi$. Let us see the peak values and the locations of the main-lobe of diffraction beams. The peak levels and locations are 17.4dB at $(\theta_s, \phi_s) \approx (15°, 67°)$ and $(15°, 293°)$, 15.3dB at $(43°, 43°)$ and $(43°, 317°)$, and 9.2dB at $(68°, 14°)$ and $(68°, 346°)$, which correspond to the diffraction orders (m, n)=$(-1, 0)$, $(-1, -1)$, $(0, 1)$, $(0, -1)$,

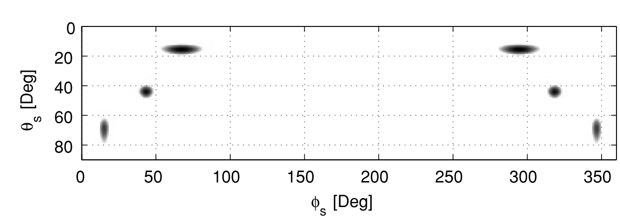

Fig. 5 Scattering cross section $\sigma(\theta_s, \phi_s|\theta_i, \phi_i)$ by the first order solution. Gray scale image. There are 6 black footprints representing the main-lobes of diffraction beams. $\theta_i = \pi/6$, $\phi_i = \pi$, $\sigma_h = 0.1\lambda$, and $\kappa = 5\lambda$.

$(1, 1)$, and $(1, 0)$, respectively. These angles almost agree with the diffraction angles calculated using the grating formula (57).

It is interesting to note that the beams are not isotropic in the (θ_s, ϕ_s) plane, even though $|\hat{G}(\mathbf{s})|^2$ is an isotropic spectrum. This is because of the transformation from $\mathbf{s}$ to (θ_s, ϕ_s) by (55). Roughly speaking, the beam width $\Delta\theta_s$ in the θ_s direction is proportional to $1/\cos\theta_s$, but the beam width $\Delta\phi_s$ in the ϕ_s direction is proportional to $1/\sin\theta_s$.

6. Conclusions

The previously introduced periodic Fourier transform concept is extended to a two-dimensional case in this paper. The relations between the periodic Fourier transform, harmonic series representation and Fourier integral representation are discussed. As a simple application of the periodic Fourier transform, the scattering of a scalar wave from a finite periodic surface with weight is studied by the small perturbation method.

Our discussions can be immediately extended to the scattering of electromagnetic waves. However, such an extension will be left for future study.

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Appendix A: Rectangular Pulse

Let $u(\mathbf{s}|C_q)$ be the rectangular pulse in the two-dimensional plane

$$u(\mathbf{s}|C_q) = u(\mu_1\mathbf{q}_1 + \mu_2\mathbf{q}_2|C_q) = \begin{cases} 1, & -1/2 \le \mu_1, \mu_2 < 1/2 \\ 0, & \text{else where} \end{cases}, \tag{A·1}$$

which satisfies

$$u^2(\mathbf{s}|C_q) = u(\mathbf{s}|C_q), \tag{A·2}$$

$$\sum_{m,n=-\infty}^{\infty} u(\mathbf{s} - \mathbf{q}(m,n)|C_q) = 1. \tag{A·3}$$

Appendix B: Properties of Periodic Fourier Transform

In this section, several properties of the periodic Fourier transform are discussed. However, for simplicity we will denote the periodic Fourier transform and its inverse by $f(\mathbf{r}) \Longleftrightarrow F(\mathbf{r}, \mathbf{s})$.

modulation and shift

If $f(\mathbf{r}) \Longleftrightarrow F(\mathbf{r}, \mathbf{s})$, then

$$f(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}} \Longleftrightarrow F(\mathbf{r}, \mathbf{s} + \mathbf{q}),$$
$$f(\mathbf{r} - \mathbf{r}_0) \Longleftrightarrow e^{-i\mathbf{s}\cdot\mathbf{r}_0} F(\mathbf{r} - \mathbf{r}_0, \mathbf{s}). \tag{A·4}$$

constant and exponential function

Since the Fourier transform of constant 1 is $(2\pi)^2\delta(\mathbf{s})$, we obtain from (37)

$$1 \Longleftrightarrow F_c(\mathbf{r}, \mathbf{s}),$$
$$F_c(\mathbf{r}, \mathbf{s}) = A_q \sum_{m,n} e^{i\mathbf{q}(m,n)\cdot\mathbf{r}} \delta(\mathbf{s} + \mathbf{q}(m,n)). \tag{A·5}$$

Using this and (A·4), we obtain

$$e^{i\mathbf{s}'\cdot\mathbf{r}} \Longleftrightarrow F_e(\mathbf{r}, \mathbf{s}),$$
$$F_e(\mathbf{r}, \mathbf{s}) = A_q \sum_{m,n} e^{i\mathbf{q}(m,n)\cdot\mathbf{r}} \delta(\mathbf{s} - \mathbf{s}' + \mathbf{q}(m,n)). \tag{A·6}$$

product of weighting function and periodic function

Let $g(\mathbf{r})$ and $f_p(\mathbf{r})$ be a weighting function and a periodic function with $f_p(\mathbf{r})$=$f_p(\mathbf{r} + \mathbf{L}_1)$=$f_p(\mathbf{r} + \mathbf{L}_2)$, respectively. If we write

$$g(\mathbf{r}) \Longleftrightarrow F_g(\mathbf{r},\mathbf{s}), \tag{A·7}$$

the product $f_p(\mathbf{r})g(\mathbf{r})$ is transformed into the product of the periodic function and the periodic Fourier transform of the weighting function

$$f_p(\mathbf{r})g(\mathbf{r}) \Longleftrightarrow f_p(\mathbf{r})F_g(\mathbf{r},\mathbf{s}), \tag{A·8}$$

indicating that a periodic factor is invariant under the periodic Fourier transform. This is an important property of the periodic Fourier transform.

inner product and Perseval's theorem

The inner product of $f(\mathbf{r})$ and $g(\mathbf{r})$ may be calculated as

$$\begin{aligned}&\int_{R^2} f(\mathbf{r})g^*(\mathbf{r})d\mathbf{r}\\ &\quad= \frac{1}{A_q}\int_{C_L} d\mathbf{r}\int_{C_q} d\mathbf{s}F(\mathbf{r},\mathbf{s})G^*(\mathbf{r},\mathbf{s}), &\text{(A·9)}\\ &\quad= \sum_{m,n=-\infty}^{\infty}\frac{A_L}{A_q}\int_{C_q} d\mathbf{s}F_{m,n}(\mathbf{s})G^*_{m,n}(\mathbf{s}), &\text{(A·10)}\end{aligned}$$

where $G(\mathbf{r},\mathbf{s})$ is the periodic Fourier transform of $g(\mathbf{r})$, and $F_{m,n}(\mathbf{s})$ and $G_{m,n}(\mathbf{s})$ are the harmonic spectra of $g(\mathbf{r})$ and $f(\mathbf{r})$,

$$\begin{bmatrix} F(\mathbf{r},\mathbf{s})\\ G(\mathbf{r},\mathbf{s})\end{bmatrix} = \sum_{m,n}\begin{bmatrix} F_{m,n}(\mathbf{s})\\ G_{m,n}(\mathbf{s})\end{bmatrix} e^{i\mathbf{q}(m,n)\cdot\mathbf{r}}. \tag{A·11}$$

Putting $g(\mathbf{r}) = f(\mathbf{r})$, we formally obtain Perseval's theorem†,

$$\begin{aligned}\int_{R^2} |f(\mathbf{r})|^2 d\mathbf{r} &= \frac{1}{A_q}\int_{C_L} d\mathbf{r}\int_{C_q} d\mathbf{s}|F(\mathbf{r},\mathbf{s})|^2\\ &= \sum_{m,n=-\infty}^{\infty}\frac{A_L}{A_q}\int_{C_q} |F_{m,n}(\mathbf{s})|^2 d\mathbf{s}. &\text{(A·12)}\end{aligned}$$

†However, Eqs. (A.8) and (A.9) in Ref. [7] were misprinted. They should be read as

$$\begin{aligned}&2\pi\int_{-\infty}^{\infty} f(x)g^*(x)dx = L\int_{-k_L/2}^{k_L/2} ds\int_{-L/2}^{L/2} F(x,s)\\ &\quad\times G^*(x,s)dx = L^2\sum_{m=-\infty}^{\infty}\int_{-k_L/2}^{k_L/2} F_m(s)G^*_m(s)ds,\\ &2\pi\int_{-\infty}^{\infty} |f(x)|^2 dx = L\int_{-k_L/2}^{k_L/2} ds\int_{-L/2}^{L/2} |F(x,s)|^2 dx\\ &\quad= L^2\sum_{m=-\infty}^{\infty}\int_{-k_L/2}^{k_L/2} |F_m(s)|^2 ds.\end{aligned}$$

Junichi Nakayama received his B.E. degree from Kyoto Institute of Technology in 1968, and his M.E. and Dr. Eng. degrees from Kyoto University in 1971 and 1982, respectively. In 1975, he joined the staff of Faculty of Engineering and Design, Kyoto Institute of Technology, where he is currently a professor of Electronics and Information Science. His research interests are electromagnetic wave theory, acoustical imaging and signal processing. Dr. Nakayama is a member of IEEE and a fellow of the Institute of Physics.