LETTER

Energy Balance Formulas in Grating Theory

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SUMMARY The energy conservation law and the optical theorem in the grating theory are discussed: the energy conservation law states that the incident energy is equal to the sum of diffracted energies and the optical theorem means that the diffraction takes place at the loss of the specularly reflection amplitude. A mathematical relation between the optical theorem and the energy conservation law is given. Some numerical examples are given for a TM plane wave diffraction by a sinusoidal surface.

key words: *energy balance formula, periodic grating, energy conservation law, optical theorem*

1. Introduction

In the theory of periodic grating, two types of energy balance formulas are known [1]–[6]. One is the energy conservation law stating that the incident energy is equal to the sum of diffracted energies. The other is the optical theorem stating that the diffraction takes place at the loss of the reflection amplitude. Many authors have applied the energy conservation law to estimate accuracy of analysis numerically. On the other hand, it seems that the optical theorem has seldom discussed, even though the theorem was given in a classical paper [6].

Recently, we have looked for relations between the scattering from a periodic surface with finite extent and the diffraction by a periodic surface with infinite extent. Then, we found in a certain case that the optical theorem could become a bridge connecting such scattering and diffraction [7]. This gives us a motivation to reconsider the optical theorem in the grating theory. This paper briefly discusses the optical theorem. Then, we present a mathematical relation between the optical theorem and the energy conservation law. We point out that the optical theorem can be used as another method to estimate accuracy of numerical analysis. Some numerical examples are given for a TM plane wave diffraction by a sinusoidal surface.

2. Diffraction by a Periodic Grating

Let us consider the TM wave diffraction from a periodic surface given by

$$z = f(x) = f(x+L), \qquad k_L = \frac{2\pi}{L} \tag{1}$$

where L is the period and k_L is the spatial angular frequency associated with L. We write the y component of the magnetic field by $\psi(x,z)$, which satisfies Helmholtz equation,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\psi(x,z) = 0, \tag{2}$$

in free space above the surface (1). Here, $k = 2\pi/\lambda$ is wavenumber and λ is wavelength. On the surface (1), the magnetic field satisfies the Neumann condition,

$$\frac{\partial}{\partial n}\psi(x,z) = 0, \qquad z = f(x), \tag{3}$$

where n is normal to the surface. We write the incident plane wave $\psi_i(x,z)$ as

$$\psi_i(x,z) = e^{-ipx}e^{-i\beta_0 z}, \qquad p = k \cdot \cos\theta_i. \tag{4}$$

Here, θ_i is the angle of incidence (See Fig. 1) and

$$\beta_m = \sqrt{k^2 - (p + mk_L)^2}, \qquad Im\,[\beta_m] \geq 0, \quad (m = 0, \pm 1, \pm 2, \cdots), \tag{5}$$

where Im stands for the imaginary part.

Since the surface deformation is periodic, the wave field $\psi(x,z)$ has the Floquet form, which we write as

$$\psi(x,z) = e^{-ipx}e^{-i\beta_0 z} + e^{-ipx}e^{i\beta_0 z} + e^{-ipx} \times \sum_{m=-\infty}^{\infty} A_m e^{-imk_L x + i\beta_m z}, \; z > \sigma, \tag{6}$$

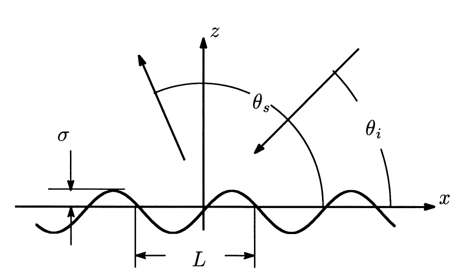

Fig. 1 Diffraction of a plane wave from a periodic surface. L is the period, σ is the surface height. The angle of incidence θ_i and a diffraction angle θ_s are measured from the positive x axis.

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where $\sigma = max\{f(x)\}$. The equation (6) may be interpreted in two ways. In the energy conservation law, (6) is considered as a sum of the incident plane wave and diffracted waves with amplitudes $\{A_m + \delta_{m,0}\}$, $\delta_{m,n}$ being Kronecker's delta. In the optical theorem below, (6) is regarded as a sum of three components: the incident plane wave, the reflected wave and diffracted waves with amplitudes $\{A_m\}$. In other words, the diffraction is defined as deviation from the sum of incident wave and the reflected wave. Note that the definition of diffracted waves is different in these two cases.

3. Energy Conservation and Optical Theorem

The energy conservation law may be written as [1]

$$\frac{1}{k}\sum_{m=-\infty}^{\infty} Re[\beta_m]|A_m + \delta_{m,0}|^2 = \frac{\beta_0}{k}, \tag{7}$$

where Re stands for the real part, the right-hand side is the incident energy falling on unit surface, and the left-hand side is the sum of diffraction energies. Note that $(1+A_0)$ is regarded as the 0 order diffraction amplitude in (7). From (7), we obtain inequalities,

$$|1 + A_0| \leq 1, \qquad -2 \leq Re[A_0] \leq 0. \tag{8}$$

We define the efficiency $\hat{P}_m$ of the m-th order diffraction and the error $\hat{e}_e$ with respect to the energy conservation as

$$\hat{P}_m = \frac{Re[\beta_m]}{\beta_0}|A_m + \delta_{m,0}|^2, \tag{9}$$

$$\hat{e}_e = \left|\sum_{m=-\infty}^{\infty} \hat{P}_m - 1\right|. \tag{10}$$

This relation (10) has been used for estimating accuracy of analysis numerically by many authors. Sometimes, (7) is called the flux density conservation, or the optical theorem. By the optical theorem, however, this paper means another formula (11) below. The optical theorem may be obtained from (7) as

$$\frac{1}{k}\sum_{m=-\infty}^{\infty} Re[\beta_m]|A_m|^2 = -2\frac{\beta_0}{k}Re[A_0], \tag{11}$$

where A_0 is regarded as the 0 order diffraction amplitude. The left-hand side in (11) is the sum of diffraction energies and hence $-2\beta_0 Re[A_0]/k$ is understood as the total diffraction energy per unit surface. Therefore, the total diffraction is proportional to the real part of the 0 order diffraction amplitude A_0. This is analogous to the forward scattering theorem stating that the total scattering cross section is proportional to the imaginary part of the forward scattering amplitude. Moreover, it is known that the total scattering cross section becomes twice of the geometrical cross section of a reflecting target much larger than wavelength [8]. If we regard $\beta_0/k = \sin\theta_i$ as the geometrical cross section of unit surface, we formally find from (11) and (8) that the total diffraction can vary from 0 to 4 times of the geometrical cross section. However, the optical theorem (11) and the forward sacttering theorem are different in physical significance.

For normalization, we introduce the relative energy of the m-th order diffraction by

$$P_m = -\frac{Re[\beta_m]}{2\beta_0 Re[A_0]}|A_m|^2, \tag{12}$$

which is the ratio of the m-th order diffraction energy to the total diffraction energy. Then, we define the error e_o with respect to the optical theorem by

$$e_o = \left|\sum_{m=-\infty}^{\infty} P_m - 1\right| = \frac{\hat{e}_e}{|2Re[A_0]|}, \tag{13}$$

which gives the mathematical relation between e_o and $\hat{e}_e$. This relation is the main result of this paper. If $|Re[A_0]| < 1/2$, e_o is larger than $\hat{e}_e$. When $|Re[A_0]| > 1/2$, however, $\hat{e}_e$ become larger than e_o. Note that $\hat{e}_e$ can be 4 times larger than e_o at most by (8). Thus, the relation (13) becomes another method to estimate accuracy of numerical solution.

4. Numerical Example

Let us consider a simple example, where the surface deformation is sinusoidal:

$$z = f(x) = \sigma\sin(k_L x). \tag{14}$$

According to the reference [2], we put

$$k = 10, \qquad L = 1, \qquad \sigma = 0.15. \tag{15}$$

Then, we numerically determined $\{A_m\}$ from $m = -10$ to 10 by use of a non-Rayleigh method in Ref. [3]. Figure 2 illustrates $-2\beta_0 Re[A_0]/k$ and A_0 against θ_i, which vary rapidly at $\theta_i \approx 27.8°$, 68.2° and 75.1° due to Wood's anomaly.

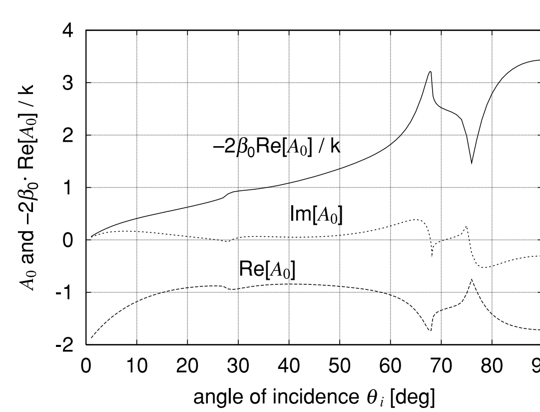

Fig. 2 A_0 and the total diffraction $-2\beta_0 Re(A_0)/k$ against angle of incidence θ_i. $L = 1$, $\sigma = 0.15$, $k = 10$.

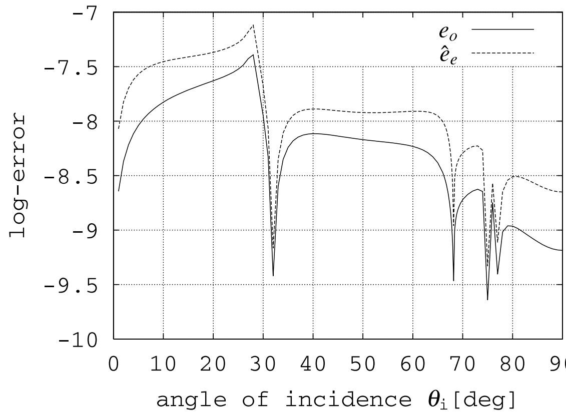

Fig. 3 $\log_{10}(e_0)$ and $\log_{10}(\hat{e}_e)$ against angle of incidence θ_i. $L = 1$, $\sigma = 0.15$, $k = 10$.

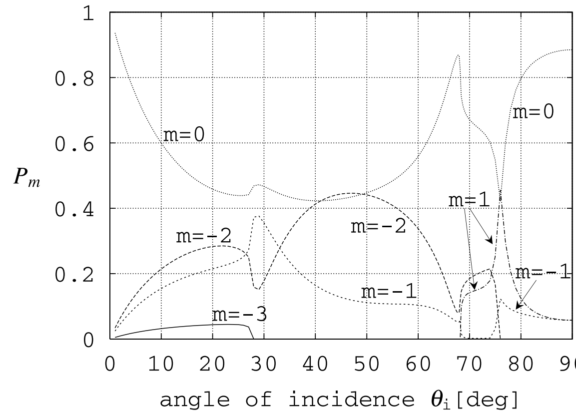

Fig. 5 Relative energy of diffraction P_m against angle of incidence θ_i. $L = 1$, $\sigma = 0.15$, $k = 10$.

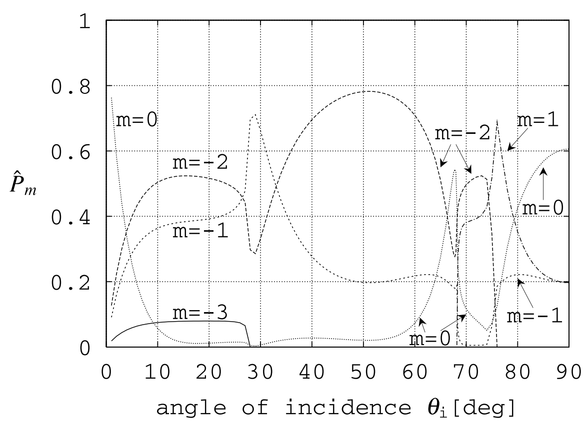

Fig. 4 Diffraction efficiency $\hat{P}_m$ against angle of incidence θ_i. $L = 1$, $\sigma = 0.15$, $k = 10$.

Figure 3 illustrates log-errors, $\log_{10}(\hat{e}_e)$ and $\log_{10}(e_o)$, which are less than -7 for any angle of incidence calculated. Since $|Re[A_0]| > 1/2$ in case of (15), $\hat{e}_e$ becomes slightly larger than e_o. From this figure, we may say again that the optical theorem (13) can be used as a method to estimate accuracy of numerical solution. The efficiency $\hat{P}_m$ against θ_i is illustrated in Fig. 4, which agrees well with other source [2]. On the other hand, Fig. 2 shows that the total diffraction energy depends on θ_i and becomes maximum at $\theta_i = 90°$. The total diffraction energy is distributed into each diffraction order at the rate of P_m, which is shown against θ_i in Fig. 5. It is interesting to see that P_m is largely different from $\hat{P}_m$ as a function of θ_i. Specially, P_0 and $\hat{P}_0$ are entirely different.

5. Conclusions

We have discussed the optical theorem in the grating theory. Then, we have presented the relation between the energy conservation law and the optical theorem.

Our discussions are restricted to a reflecting grating for TM case but can be immediately extended to TE case. Extension to a transmitting and lossy case may be possible, where the optical theorem is expected to be a relation between the 0 order diffraction amplitude and the sum of diffraction energies and absorption energy. However, we are interested in the optical theorem (11), because the theorem could be a bridge connecting the diffraction by a periodic grating with infinite extent and the scattering from a periodic grating with finite extent [7]. Theoretical and numerical studies on such connection are also left for future study.

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