LETTER

# Low Grazing Scattering from Periodic Neumann Surface with Finite Extent

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**SUMMARY** This paper deals with the scattering of transverse magnetic (TM) plane wave by a perfectly conductive surface made up of a periodic array of finite number of rectangular grooves. By the modal expansion method, the total scattering cross section $p_c$ is numerically calculated for several different numbers of grooves. It is then found that, when the groove depth is less than wavelenght, the total scattering cross section $p_c$ increases linearly proportional to the corrugation width $W$. But an exception takes place at a low grazing angle of incidence, where $p_c$ is proportional to $W^\alpha$ and the exponent $\alpha$ is less than 1. From these facts, it is concluded that the total scattering cross section $p_c$ must diverge but $p_c/W$ the total scattering cross section per unit surface must vanish at a low grazing limit when the number of grooves goes to infinity.

***key words:*** *low-grazing-angle scattering, periodic array of rectangular grooves. total scattering cross section, total scattering cross section per unit surface*

## 1. Introduction

Low grazing scattering from a rough surface is an important problem in the study of sea observation by a ground based radar [1]. This is an interesting problem, because singularity appears in the case of the Neumann surface. When the rough surface is infinite in extent, only reflection but no diffraction or no scattering take place at a low grazing limit of incidence (LGLI). Such singular behavior is predicted analytically in the case of a homogenous random surface with slight roughness [2]–[4]. It was also demonstrated analytically and numerically for periodic Neumann surfaces [5], [6]. On the other hand, Tatarski and Charnotskii [7] pointed out theoretically that the scattering takes place at LGLI, if the rough surface is finite in extent. This fact is supported by numerical calculations [8].

One says only reflection takes place but the other insists the scattering may occur at LGLI. Obviously, there is a wide gap in these discussions but it seems that no works have been carried out to bridge over the wide gap. To bridge over the gap, we present a simple example. We deal with the scattering from a periodic array of finite number of rectangular grooves as is shown in Fig. 1. By the modal expansion method [9]–[11], the total scattering cross section is numerically calculated for several different numbers of grooves. Generally, the total scattering cross section $p_c$ increases linearly proportional to the corrugation width $W$. However, we newly find that an exception takes place at LGLI, where $p_c$ is roughly proportional to $W^\alpha$ with the exponent $\alpha$ less than 1. From this fact, it is concluded that the total scattering cross section $p_c$ must diverge but the total scattering cross section per unit surface $p_c/W$ must vanish at LGLI as the number of grooves goes to infinity. Thus, the total diffraction cross section vanishes at LGLI, because it is given as the total scattering cross section per unit surface.

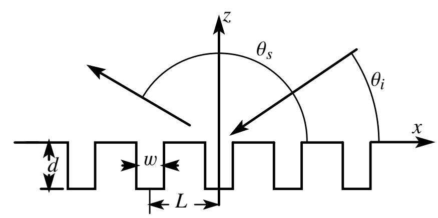


**Fig. 1** Scattering of a TM plane wave from a finite number of rectangular grooves. $\theta_i$ is the angle of incidence, $w$ and $d$ are the width and depth of a groove. $L$ is the period.

## 2. Modal Expansion Method

Let us consider the scattering of a TM plane wave by a finite number of rectangular grooves in Fig. 1. We write the periodic surface with finite extent as

$$z = f(x) = -d \sum_{g=-N_g}^{N_g} u(x - gL), \tag{1}$$

$$u(x) = \begin{cases} 1, & |x| \le w/2 \\ 0, & |x| > w/2 \end{cases}, \tag{2}$$

where $d$ is the depth of a groove, $L$ is the period and $(2N_g+1)$ is the total number of grooves. $u(x)$ is the rectangular pulse with the pulse width $w$. To simplify notation, we put

$$W = (2N_g + 1)L, \quad k_L = \frac{2\pi}{L}, \quad k_w = \frac{\pi}{w}, \tag{3}$$

where $W$ is the equivalent width of corrugation.

We denote the $y$ component of the magnetic filed by $\psi(x, z)$, which satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\psi(x, z) = 0, \tag{4}$$

above the surface (1) and Neumann condition on the surface (1). Here, $k = 2\pi/\lambda$ is wave number and $\lambda$ is wavelength. We denote the magnetic field above the $x$ axis by $\psi_1(x, z)$,

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which may be written as

$$\psi_1(x,z) = e^{-ipx-i\beta(p)z} + e^{-ipx+i\beta(p)z} + \psi_s(x,z), \tag{5}$$

where the first term on the right hand side is the incident plane wave, the second the reflected wave and the third the scattered wave due to the corrugation. Here, $p$ and $\beta(p+s)$ are defined by

$$p = k\cos(\theta_i), \tag{6}$$

$$\beta(p+s) = \sqrt{k^2-(p+s)^2}, \tag{7}$$

$$Re[\beta(p+s)] \geq 0, \quad Im[\beta(p+s)] \geq 0, \tag{8}$$

where $\theta_i$ is the angle of incidence, measured from the positive $x$ axis as shown in Fig. 1. The Fourier spectrum of the scattered wave has singularities but the angular spectrum is always finite [8]. In terms of the angular spectrum $A_\beta(s)$, we write $\psi_s(x,z)$ as

$$\psi_s(x,z) = \int_{-k_B}^{k_B} \frac{A_\beta(s)}{\beta(p+s)} e^{-i(p+s)x+i\beta(p+s)z} ds, \tag{9}$$

where $k_B$ is a sufficiently large bandwidth. On the other hand, we denote the magnetic field inside the grooves by $\psi_2(x,z)$, which may be represented by a sum of guided modes as [9]–[11]

$$\psi_2(x,z) = \sum_{g=-N_g}^{N_g} u(x-gL)e^{-ipgL} \sum_{m=0}^{N_{gm}} E_m^{(g)} \times \cos[mk_w(x+w/2-gL)]\cos[\gamma_m(z+d)], \tag{10}$$

$$\gamma_m = \sqrt{k^2-m^2k_w^2}. \tag{11}$$

Here, $N_{gm}$ is a truncation number of guided modes, $E_m^{(g)}$ the amplitude of the $m$th guided mode in the $g$th groove and $\gamma_m$ is the propagation constant of the $m$th guided mode. When $k^2 < m^2k_w^2$, $\gamma_m$ becomes pure imaginary.

Next, we write the optical theorem as [8],

$$p_c = p_{inc}, \tag{12}$$

$$p_c = -\frac{4\pi}{k} Re[A_\beta(0)], \tag{13}$$

$$p_{inc} = \frac{2\pi}{k} \int_{-\infty}^{\infty} Re[\beta(p+s)] \left| \frac{A_\beta(s)}{\beta(p+s)} \right|^2 ds. \tag{14}$$

The optical theorem (12) states that the total scattering cross section† $p_{inc}$ is equal to $p_c$ the loss of the specularly reflected component. Because of (12), however, we will call $p_c$ the total scattering cross section below. We may define the error $E_{rr}$ with respect to the optical theorem (12) as

$$E_{rr} = |p_c - p_{inc}|/p_c. \tag{15}$$

Let us determine $A_\beta(s)$ and $E_m^{(g)}$ by the boundary conditions on the $z=0$ plane, which are

$$\sum_{g=-N_g}^{N_g} u(x-gL)\left[\psi_1(x,0) - \psi_2(x,0)\right] = 0, \tag{16}$$

$$\left[\frac{\partial\psi_1(x,z)}{\partial z} - \frac{\partial\psi_2(x,z)}{\partial z}\right]_{z=0} = 0. \tag{17}$$

After some manipulations, we obtain an equation system for the $(2N_g+1)\times(N_{gm}+1)$-vector $\{E_{m_1}^{(g_1)}\}$ as

$$\frac{w}{2}(1+\delta_{m_1,0})\cos(\gamma_{m_1}d)E_{m_1}^{(g_1)} + \sum_{g_2=-N_g}^{N_g}\sum_{m_2=0}^{N_{gm}} K_{(m_1,m_2)}^{(g_1,g_2)} \times e^{ipL(g_1-g_2)}\gamma_{m_2}\sin(\gamma_{m_2}d)E_{m_2}^{(g_2)} = 2c_{m_1}(-p), \tag{18}$$

which holds for $g_1 = -N_g, -N_g+1, \cdots, N_g$ and $m_1 = 0, 1, \cdots, N_{gm}$. Here,

$$K_{(m_1,m_2)}^{(g_1,g_2)} = \int_{-k_B}^{k_B} \frac{c_{m_1}(-s)c_{m_2}(s)}{2\pi i\beta(s)} e^{iLs(g_2-g_1)} ds, \tag{19}$$

$$c_m(q) = \int_{-\infty}^{\infty} u(x)\cos[mk_w(x+w/2)]e^{iqx}dx. \tag{20}$$

From $E_m^{(g)}$, we obtain the angular spectrum $A_\beta(s)$ as

$$2\pi i A_\beta(s) = -\sum_{g=-N_g}^{N_g}\sum_{m=0}^{N_{gm}} c_m(p+s)e^{isgL}\gamma_m\sin(\gamma_m d)E_m^{(g)}, \tag{21}$$

in terms of which we will calculate $p_c$ and $p_{inc}$.

## 3. Relation with Perfectly Periodic Case

When $N_g$ goes to infinity and the surface (1) becomes perfectly periodic, the wave field may have the Floquet form, which we write [12]

$$\psi_1(x,z) = e^{-ipx-i\beta(p)z} + e^{-ipx+i\beta(p)z} + \sum_{m=-N_{fm}}^{N_{fm}} A_m e^{-i(p+mk_L)x+i\beta(p+mk_L)z}, \tag{22}$$

where $N_{fm}$ is the truncation number of Floquet modes, $A_0+1$ is the reflection coefficient and $A_m$ ($m \neq 0$) is the diffraction amplitude of the $m$th Floquet mode.

The optical theorem may be written as [12],

$$p_c^{(g)} = p_{inc}^{(g)}, \tag{23}$$

$$p_c^{(g)} = -2\frac{\beta(p)}{k}Re[A_0], \tag{24}$$

$$p_{inc}^{(g)} = \frac{1}{k}\sum_{m=-\infty}^{\infty} Re[\beta(p+mk_L)]|A_m|^2. \tag{25}$$

Here, the optical theorem (23) means that the total diffraction cross section $p_{inc}^{(g)}$ is equal to $p_c^{(g)}$ the loss of specularly reflection amplitude. Note that $p_c^{(g)} = 0$ means that

†Because $p_{inc}$ is linearly proportional to the total power of scattering, it was called the total power of scattering in Ref. [8]. In this paper, however, we call $p_{inc}$ the total scattering cross section. We note that $p_{inc}$ and $p_c$ are length in physical dimension but $p_{inc}^{(g)}$ in (25) is dimensionless.

no diffraction takes place, because $p_c^{(g)} = p_{inc}^{(g)} = 0$ implies that any propagation Floquet mode with $Re[\beta(p + mk_L)] > 0$ has a vanishing amplitude. However, it is known [6] that $A_m$ ($m \neq 0$) vanishes and $A_0$ approaches to $-2$ at LGLI with $\theta_i \to 0$. Thus, $p_c^{(g)}$ is proportional to $\sin(\theta_i)$ and vanishes as $\theta_i \to 0$, as is illustrated later.

On the other hand, the total scattering cross section $p_c$ increases proportional to $W = (2N_g + 1)L$ and diverges when $N_g \to \infty$. This means that the total scattering cross section has no physical significance in the periodic case but the densities such as $p_c/W$ can be well defined in the limit $N_g \to \infty$. Since $p_c^{(g)}$ and $p_{inc}^{(g)}$ are defined as densities per unit surface, we physically expect

$$\lim_{W\to\infty} \frac{p_c}{W} = \lim_{N_g\to\infty} \frac{p_c}{(2N_g + 1)L} = p_c^{(g)}. \tag{26}$$

As is seen below, this gives a solution to our contradiction such that the scattering takes place but no diffraction occurs at LGLI.

## 4. Numerical Example

For numerical calculation, we set $k_B$ and $N_{fm}$ in terms of $N_{gm}$ the truncation number of guided modes as

$$k_B = (N_{gm} + 1)k_w, \quad N_{fm} = \left[\frac{k_B}{k_L} + \frac{1}{2}\right]_r, \tag{27}$$

where $[\cdot]_r$ means round down operation. Roughly speaking, $\psi_s(x, z)$, $\psi_2(x, z)$ and (22) have the same bandwidth under (27). We put

$$L = 1.5\lambda, \quad w = 0.75\lambda, \quad d = 0.2\lambda, \tag{28}$$

from which $\theta_i \approx 0°$ and 70.529° become critical angles of incidence and only the lowest two guided modes become propagating. To obtain a highly accurate solution, it is desirable to set $N_{gm}$ as large as possible. However, we put $N_{gm} = 5$ and $N_{fm} = 6$ in calculations below, because of the limitation of computer resources. We numerically solved (18) for $\theta_i$ from 0.00001° to 90°, and for $(2N_g + 1) = 11$, 33, 99, 297 and 891, where $E_{rr}$ by (15) is always less than $10^{-4}$.

Figure 2 illustrates $p_c$ against $\theta_i$, where $p_c$ changes rapidly at $\theta_i \approx 70.5°$ due to Wood's anomaly. $p_c$ increases proportional to $(2N_g + 1)$. However, $p_c$ at $\theta_i \approx 0$ increases quite slowly. To see this, we illustrate $p_c$ against $(2N_g + 1)$ at $\theta_i = 0.00001°$ for $d = 0.1\lambda$, $0.2\lambda$, $0.4\lambda$, $0.6\lambda$, $0.8\lambda$ and $1.0\lambda$ in Fig. 3. When $d = 0.2\lambda$, $\log(p_c/\lambda)$ is almost linearly proportional to $\log(2N_g + 1)$, but such a linear proportion can not be observed for other values of $d$ in Fig. 3. For a sufficiently large $(2N_g + 1)$, however, we expect that $p_c/\lambda = a(2N_g + 1)^\alpha$ holds asymptotically. We determined the exponent $\alpha$ from the data at $(2N_g + 1) = 297$ and 891. Then, we found $\alpha = 0.4597$ at $d = 0.1\lambda$, 0.5011 at $0.2\lambda$, 0.5281 at $0.4\lambda$ , 0.5867 at $0.6\lambda$, 0.4894 at $0.8\lambda$ and 0.5539 at $1.0\lambda$ for $\theta_i = 0.00001°$. It is important to say that the exponent $\alpha$ is always less than 1[†]. Thus, we may conclude $p_c$ must diverge but $p_c/W$ must vanish as $W = (2N_g + 1)L$ goes to infinity.

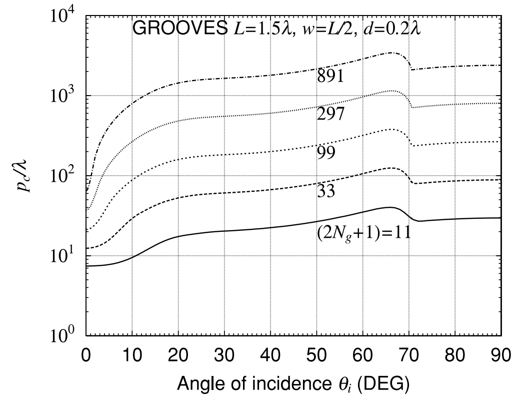


**Fig. 2** $p_c/\lambda$ against $\theta_i$ the angle of incidence. $L = 1.5\lambda$, $w = L/2$, $d = 0.2\lambda$.

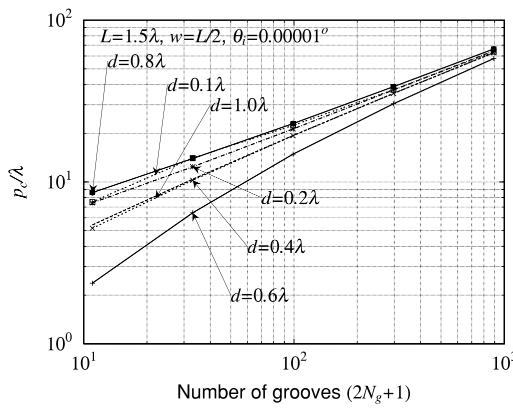


**Fig. 3** $p_c/\lambda$ against $(2N_g + 1)$. $L = 1.5\lambda$, $w = L/2$, $\theta_i = 0.00001°$.

Figure 4 illustrates $p_c/W$ and $p_c^{(g)}$ against $\theta_i$, where $p_c^{(g)}$ vanishes as $\theta_i \to 0$ as is described above. Even when $(2N_g + 1) = 11$, $p_c/W$ is close to $p_c^{(g)}$ except for $\theta_i < 20°$ and $60° < \theta_i < 75°$. When $(2N_g + 1) = 891$, $p_c/W$ becomes almost equal to $p_c^{(g)}$ for any $\theta_i > 2°$. This means that, when $W$ is sufficiently large, (26) holds practically and $p_c$ is proportional to $(2N_g + 1)$ except for a small value of $\theta_i$. In Fig. 4(B), we see that $p_c/W$ at LGLI decreases as $(2N_g + 1)$ increases.

## 5. Conclusions

We studied numerically the scattering of a TM plane wave by a periodic array of finite number of rectangular grooves. We found that the total scattering cross section $p_c$ increases linearly proportional to the corrugation width $W$, when the number of grooves is large enough and when the angle of

[†]Numerical examples give $\alpha = 0.5428$ for $d = 5.1\lambda$ and $\alpha = 0.4992$ for $d = 6.2\lambda$. Thus, it seems that $\alpha < 1$ holds generally even for $d > \lambda$. When $\gamma_{m_1}$ and $\gamma_{m_2}$ are pure imaginary, however, $\cos(\gamma_{m_1}d)$ and $\sin(\gamma_{m_2}d)$ in (18) increase exponentially for a large $d$. This makes it difficult to obtain a highly accurate numerical solution from (18) for $d \gg \lambda$. Thus, the range of $d$ within which $\alpha < 1$ holds is not clear at present.

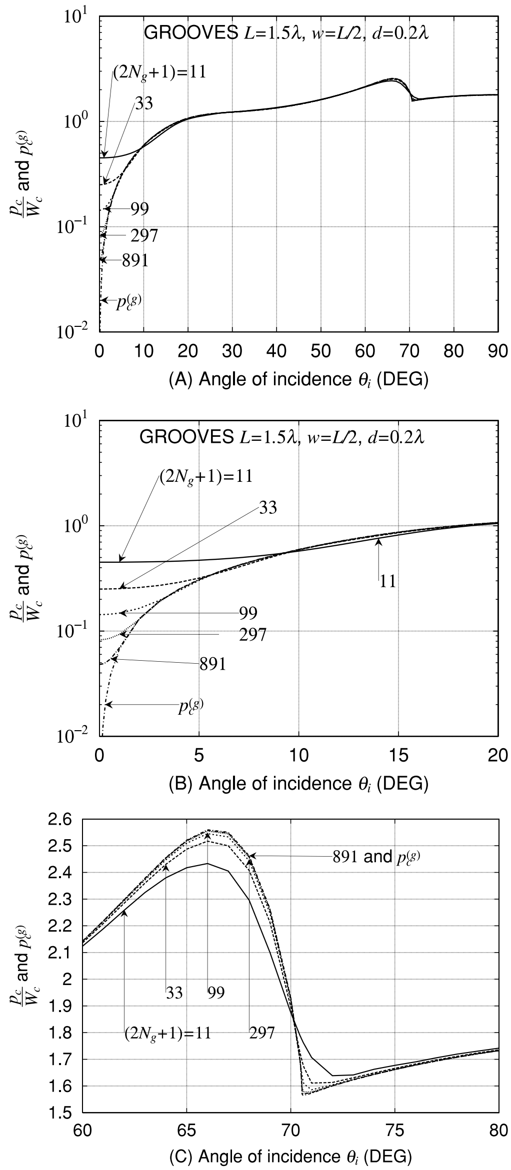


**Fig. 4** $p_c/W$ and $p_c^{(g)}$ against $\theta_i$. (B) and (C) are enlarged figures. $L = 1.5\lambda$, $w = L/2$, $d = 0.2\lambda$.

incidence is not low grazing. At a low grazing angle of incidence, however, we found a remarkable result such that $p_c$ is roughly proportional to $W^{\alpha}$ with the exponent $\alpha$ less than 1. This suggests that the total scattering cross section $p_c$ diverges but the total scattering cross section per unit surface $p_c/W$ vanishes at a low grazing limit when the number of grooves goes to infinity. Thus, the total diffraction cross section vanishes at LGLI, because it is given as $p_c/W$ the total scattering cross section per unit surface. From these facts, we may conclude that the scattering must take place from a finite periodic surface and the diffraction by a perfectly periodic surface must not occur at LGLI.

We discussed a numerical solution for the case of a finite array of rectangular grooves, where the maximum number of grooves was limited to $(2N_g + 1) = 891$ due to the limitation of computer resources. Also our discussions were restricted to a shallow case with $d \leq \lambda$. However, we expect that, even for deep grooves with $d \gg \lambda$, the total scattering cross section at LGLI diverges but the total scattering cross section per unit surface vanishes as the corrugation width goes to infinity. However, mathematical and numerical discussions on our expectation are left for future study.

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