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# A Method for Construction of Wavelet Systems

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**Abstract**

In this report we give a method for construction of wavelet system from MRA, which is different from Daubechies-Heller method. In addition we set several conditions in order to form the smooth wavelet under various parameters.

**Key Words:** *n-MRA; symbol matrix; moment condition; zero point condition.*

## 1. Introduction

It is well-known that "good" wavelet systems are made up through a multiresolution analysis (MRA) which was formulated by Meyer and Mallat.

The method for construction of wavelet systems with compact support treated by Daubechies and Heller is as follows:[4),5)]

(1) Solve the functional equation: $\sum_{k=0}^{n-1}\left|m_0(\omega^k z)\right|^2 = 1$ with $m_0(1) = 1$, where $m_0(z)$ is called a symbol of scaling function and $\omega = \exp(-2\pi i/n)$.

(2) Obtain the wavelet sequence from the scaling sequence which consists of the coefficients of the symbol $m_0(z)$.

(3) Obtain a wavelet system from the wavelet sequence.

In this research, we aim to obtain a scaling function and its associated wavelet system at the same time from a different method than that given by Daubechies-Heller. We carried it out by construction of a certain unitary matrix which is deeply concerned with symbols of scaling function and wavelets.

## 2. Construction of Wavelet System

**Definition2.1.** *An n-multiresolution analysis (n-MRA) for a natural number $n \geq 2$ is a collection $\{V_j\}_{j\in \mathbf{Z}}$ of subspaces of $L^2(\mathbf{R})$ such that*

(1) $V_j \subset V_{j+1}$ for all $j \in \mathbf{Z}$

(2) $\bigcap_{j\in \mathbf{Z}} V_j = \{0\}$

(3) $\overline{\bigcup_{j\in \mathbf{Z}} V_j} = L^2(\mathbf{R})$

(4) $f(x) \in V_j \Longleftrightarrow f(nx) \in V_{j+1}$

(5) *There exists a function* $\phi \in V_0$ *such that* $\{\phi(x-k)\}_{k\in \mathbf{Z}}$ *makes an orthonormal basis in* $V_0$.

Function $\phi$ is called a scaling function of $n$-MRA.

Here, we aim to choose wavelet functions $\psi^1, \dots, \psi^{n-1} \in V_1$ of $n$-MRA such that

$$\begin{cases} \left(\phi(x-k), \psi^s(x-l)\right) = 0 \\ \left(\psi^s(x-k), \psi^r(x-l)\right) = \delta_{s,r}\delta_{l,k}, \end{cases}$$

for $k, l \in \mathbf{Z}, r, s = 1, \dots, n-1$.

As is well known we can write

$$\hat{\phi}(\xi) = m_0\left(e^{-\frac{i\xi}{n}}\right)\hat{\phi}\left(\frac{\xi}{n}\right)$$

$$\hat{\psi}^s(\xi) = m_s\left(e^{-\frac{i\xi}{n}}\right)\hat{\phi}\left(\frac{\xi}{n}\right), \qquad (s = 1, \dots, n-1)$$

where $\hat{}$ means the Fourier transform and $m_0(z)$ is the symbol of the scaling function, and $m_s(z)$ $(s = 1, \dots, n-1)$ are the symbols of the wavelet functions.

Here define the symbol matrix and the symbol vector as follows:

$$M(z) := \left(\vec{m}(z)\ \vec{m}(\omega z)\ \dots\ \vec{m}(\omega^{n-1}z)\right)$$

$$\vec{m}(z) := {}^t\left(m_0(z) m_1(z) \dots m_{n-1}(z)\right),$$

where $\omega := \exp\left(-2\pi i/n\right)$. It should be noted that the symbol matrix $M(z)$ must be unitary.

In order to construct the symbol matrix $M(z)$, we rewrite the symbols as follows:

$$\begin{aligned} m_s(z) &= \frac{1}{\sqrt{n}} \sum_{k\in \mathbf{Z}} q_k^s z^k \\ &= \sum_{l=0}^{n-1} a_l^s(z^n) z^l, \qquad (s = 0, 1, \dots, n-1) \end{aligned}$$

where $z := e^{-i\xi}$, $\{q_k^0; k \in \mathbf{Z}\}$ is the scaling sequence, $\{q_k^s : k \in \mathbf{Z}\}$ $(s = 1, \dots, n-1)$ are the wavelet sequences and $a_l^s(z^n) := n^{-1/2} \sum_{j\in \mathbf{Z}} q_{nj+l}^s z^{nj}$.

Let

$$B(z) := \left(\vec{b}(z)\ \vec{b}(\omega z)\ \dots\ \vec{b}(\omega^{n-1}z)\right)$$

$$\vec{b}(z) := {}^t\left(1\ z\ \dots\ z^{n-1}\right).$$

**Theorem 2.1.** *The symbol matrix* $M(z)$ *is unitary if and only if* $\sqrt{n}\,A(z^n)$ *is unitary, where*

$$A(z^n) := \begin{pmatrix} a_0^0(z^n) & \cdots & a_{n-1}^0(z^n) \\ \vdots & \ddots & \vdots \\ a_0^{n-1}(z^n) & \cdots & a_{n-1}^{n-1}(z^n) \end{pmatrix}$$

*Proof:* It holds that

$$M(z) = A(z^n) B(z),$$

which completes the proof since $B(z)/\sqrt{n}$ is unitary. (Q.E.D.)

Hence we see that it suffices to construct a symbol matrix $M(z)$ in order to obtain wavelet system of $n$-MRA. If we can construct $\sqrt{n}\,A(z^n)$ so that it becomes unitary, we can get a symbol vector $\vec{m}(z)$ through the symbol matrix $M(z) = A(z^n)B(z)$. That is, both the scaling and wavelet functions are decided at the same time.

Here, suppose $\sqrt{n}\,A(z^n)$ is unitary. We then get the following equation from Taylor expansion around $z^n = 1$:

$$\sqrt{n}\,A(z^n) = H + \sum_{k \in \mathbf{N}} (z^n - 1)^k V_k.$$

**Proposition 2.1.** *The matrix $H$ is an orthogonal matrix and its first row is given as $\left(1/\sqrt{n}, \ldots, 1/\sqrt{n}\right)$. The first row of $A(1)$ is given as $(1/n, \ldots, 1/n)$.*
*Proof:* Since $M(z)$ is unitary and $m_0(1) = 1$, it should be noted that $m_0(\omega) = \ldots = m_0(\omega^{n-1}) = 0$. (Q.E.D.)

The above matrix $H$ is called Haar matrix. Fixing a Haar matrix $H$, we have the following representation:

$$\sqrt{n}\,A(z^n) = \left(E + \sum_{k \in \mathbf{N}} (z^n - 1)^k U_k\right) H, \qquad (U_k := H^{-1} V_k)$$

where $E$ is the unit matrix. So we should give $\{U_1, U_2, \ldots\}$ so that $E + \sum_{k \in \mathbf{N}} (z^n - 1)^k U_k$ becomes unitary.

The following theorem[4)] plays an essential role.

**Theorem 2.2.** *(Vaidyanathan) Let $l \in \mathbf{N}$. The matrix $E + \sum_{k=1}^{l} (z^n - 1)^k U_k$ $(U_l \neq O)$ is unitary if and only if there exists $l$ unit vectors $\vec{p}_1, \ldots, \vec{p}_l \in \mathbf{R}^n$ such that*

$$E + \sum_{k=1}^{l} (z^n - 1)^k U_k = \prod_{k=1}^{l} \left(E + (z^n - 1) \vec{p}_k\, {}^t\vec{p}_k\right).$$

Hence we know that we can make a symbol vector $\vec{m}(z)$ whenever a set of real unit vectors $\{\vec{p}_1, \ldots, \vec{p}_l\}$ is given. As a special case, we can choose $\sqrt{n}\,A(z^n) = H$. The wavelet of this case is called Haar wavelet.

## 3. Wavelet with Moment Condition

It is important to make wavelet system which satisfies the moment condition in order to construct smooth wavelet system.

For a fixed Haar matrix $H$, let

$$\begin{aligned}\vec{w}(z) &= {}^t(w_0(z)\ w_1(z)\ \ldots\ w_{n-1}(z)) \\ &:= H\vec{b}(z)/\sqrt{n}.\end{aligned} \tag{3.1}$$

Let $\vec{h}_{j-1}$ $(j=1,\ldots,n)$ be the $j$-th row vector of $H$, then we have

$$w_{j-1}(z) = \langle \vec{h}_{j-1}, \vec{b}(z)/\sqrt{n}\rangle, \qquad (j=1,\ldots,n)$$

where for any two vectors $\vec{x} = {}^t(x_0\ x_1\ \ldots\ x_{n-1})$ and $\vec{y} = {}^t(y_0\ y_1\ \ldots\ y_{n-1})$, we put

$$\langle \vec{x}, \vec{y}\rangle := {}^t\vec{x}\vec{y} = \sum_{k=0}^{n-1} x_k y_k.$$

Particularly, if $\vec{x}$ and $\vec{y}$ belong to $\mathbf{R}^n$ then $\langle \vec{x}, \vec{y}\rangle$ coincides with the canonical inner product in $\mathbf{R}^n$.

From $\langle \vec{h}_{j-1}, n^{-1/2}\vec{b}(1)\rangle = 0$ for $j \geq 2$, we know that $w_{j-1}(z)$ is factored into

$$w_{j-1}(z) = a(z)\gamma_{j-1}(z), \qquad (j=2,3,\ldots,n) \tag{3.2}$$

where

$$a(z) := \frac{z-1}{2}.$$

Here we define

$$\vec{\gamma}(z) := {}^t(\gamma_1(z)\ \gamma_2(z)\ \ldots\ \gamma_{n-1}(z)).$$

**Definition 3.1.** *A symbol vector $\vec{m}(z)$ is said to satisfy the moment condition of order $N$ if*

$$m_j(z) \equiv 0 \ \left(\mathrm{mod}\ a(z)^{N+1}\right). \qquad (j=1,\ldots,n-1)$$

It should be remarked that from eq. (3.2) any symbol vector satisfies the moment condition of order 0.

As is well known, the following proposition holds.

**Proposition 3.1.** *The following 1, 2 and 3 are equivalent to one another.*

(1) *The symbol vector $\vec{m}(z)$ satisfies the moment condition of order $N$.*

(2) *Let $\psi^j(x)$ be the wavelet function with the symbol $m_j(z)$. Then it holds that*

$$\int_{\mathbf{R}} x^k \psi^j(x)\,dx = 0. \qquad (k=0,1,\ldots,N,\ \ j=1,\ldots,n-1)$$

(3) *It holds that*

$$m_0(z) \equiv 0 \quad \left(\mathrm{mod}\ w_0(z)^{N+1}\right).$$

From the definition, we see that a symbol vector $\vec{m}(z)$ satisfies the moment condition of order $N$ if and only if it holds that

$$m_j^{(k)}(1) = 0. \qquad (k = 0, 1, \dots, N, \quad j = 1, \dots, n-1)$$

So the symbol vector $\vec{m}(z)$ with the moment condition of order $N$ is restricted by the above $(n-1)N$ conditions. Then we aim to give $N$ real unit vectors $\vec{p}_1, \dots, \vec{p}_N$ so that the following vector

$$\vec{m}(z) = \prod_{k=1}^{N} \left( E + (z^n - 1)\vec{p}_k {}^t\vec{p}_k \right) \vec{w}(z) \tag{3.3}$$

satisfies the moment condition of order $N$.

Here we expand $\vec{m}(z)$ by $z^n - 1$. Then since $a(z)\,w_0(z) = (z^n - 1)/2n$, we have

$$\vec{m}(z) = \vec{w}(z) + \sum_{k=1}^{N} \left( a(z)\,w_0(z) \right)^k \vec{c}_k(z) \tag{3.4}$$

$$\vec{c}_k(z) = (2n)^k \sum_{*} \left( \prod_{j=1}^{k-1} \langle \vec{p}_{h(j)}, \vec{p}_{h(j+1)} \rangle \right) \langle \vec{p}_{h(k)}, \vec{w}(z) \rangle\, \vec{p}_{h(1)}, \qquad (k = 1, \dots, N)$$

where $\sum_*$ means the summation of all $\{h(1), \dots, h(k)\}$ which satisfies the condition: $1 \le h(1) < h(2) < \cdots < h(k) \le N$, and we for convenience put $\prod_{j=1}^{0}(\cdots) = 1$. The components of $\vec{c}_k(z)$ are polynomials of a degree less than $n-1$ since the degree of $\vec{w}(z)$ is at most $n-1$.

Here we put $\vec{p}_k = {}^t(q_k\ {}^t\vec{r}_k)$ ($q_k \in \mathbf{R}$, $\vec{r}_k \in \mathbf{R}^{n-1}$, $k = 1, \dots, N$) and define

$$P := (\vec{p}_1\ \vec{p}_2 \dots \vec{p}_N) \in \mathbf{R}^{n \times N}$$
$$\vec{q} := {}^t(q_1\ q_2 \dots q_N) \in \mathbf{R}^N$$
$$R := (\vec{r}_1\ \vec{r}_2 \dots \vec{r}_N) \in \mathbf{R}^{(n-1) \times N}$$

$$S := \begin{pmatrix} 0 & & & & \\ \langle \vec{p}_2, \vec{p}_1 \rangle & 0 & & 0 & \\ \langle \vec{p}_3, \vec{p}_1 \rangle & \langle \vec{p}_3, \vec{p}_2 \rangle & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \langle \vec{p}_N, \vec{p}_1 \rangle & \langle \vec{p}_N, \vec{p}_2 \rangle & \cdots & \langle \vec{p}_N, \vec{p}_{N-1} \rangle & 0 \end{pmatrix} \in \mathbf{R}^{N \times N}.$$

**Lemma 3.1.** *The first component of $\vec{c}_k(z)$ is given by*

$$(2n)^k \left( \langle \vec{q}, S^{k-1}\vec{q} \rangle\, w_0(z) + a(z) \langle RS^{k-1}\vec{q}, \vec{\gamma}(z) \rangle \right). \qquad (k = 1, \dots, N)$$

Here let

$$\eta_k := \langle \vec{q}, S^k \vec{q} \rangle$$
$$u_k(z) := \langle RS^k \vec{q}, \vec{\gamma}(z) \rangle,$$

then from eq. (3.4) and Lemma 3.1 we have

$$m_0(z) = w_0(z) \left\{ 1 + \sum_{k=1}^{N} (2na(z))^k w_0(z)^{k-1} (\eta_{k-1} w_0(z) + a(z) u_{k-1}(z)) \right\}.$$

So the symbol vector $\vec{m}(z)$ satisfies the moment condition of order $N$ if and only if there exists $N$ polynomials $g_k(z)$ ($k = 1, \ldots, N$, deg $g_k(z) = k$) such that

$$w_0(z) g_k(z) = g_{k-1}(z) + (2n)\eta_{k-2} a(z)^{k-1} + (2n)^k a(z)^{k+1} u_{k-1}(z), \qquad (k = 1, \ldots, N) \quad (3.5)$$

where we put $g_0(z) = 1$ and $S^{-1} = O$ for convenience.

**Lemma 3.2.** *Let $f(z)$ be a polynomial whose degree is at most $k \in \mathbf{N} \cup \{0\}$. Then there exist two polynomials $g(z)$ (deg $g(z) = k + 1$) and $h(z)$ (deg $h(z) = n - 2$) uniquely such that*

$$f(z) + a(z)^{k+2} h(z) = w_0(z) g(z).$$

**Lemma 3.3.** *$\{w_0(z), w_1(z), \ldots, w_{n-1}(z)\}$ is linearly independent in vector space spanned by polynomials of a degree of at most $n - 1$.*

*Proof:* It follows immediately from eq. (3.1). (Q.E.D.)

**Lemma 3.4.** *Let $f(z) = \sum_{k=0}^{n-2} f_k z^k$. Then we can find a vector $\vec{f}^* = {}^t(f_1^* \ldots f_{n-1}^*)$ uniquely which satisfies $f(z) = \langle \vec{f}^*, \vec{\gamma}(z) \rangle$.*

*Proof:* Since the degree of $a(z)f(z)$ is $n - 1$, we can find a set of coefficients $\{f_0^*, f_1^*, \ldots, f_{n-1}^*\}$ uniquely from Lemma 3.3 such that $a(z)f(z) = \langle \vec{f}^*, \vec{w}(z) \rangle$. Then substituting $z = 1$ and recalling $w_k(z) = a(z)\gamma_k(z)$ ($k = 1, \ldots, n - 1$) and $w_0(1) = 1$, we have $f_0^* = 0$ and

$$\begin{pmatrix} f_0^* \\ \\ \vec{f}^* \\ \\ \end{pmatrix} = H \begin{pmatrix} -f_0 \\ f_0 - f_1 \\ \vdots \\ f_{n-3} - f_{n-2} \\ f_{n-2} \end{pmatrix}$$

(Q.E.D.)

**Lemma 3.5.** *The following relation holds:*

$$w_0(z) = 1 + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2)\cdots(n-k)}{(k+1)!} a(z)^k.$$

*Proof:* It holds from Taylor expansion that

$$w_0(z) = 1 + \sum_{k=1}^{n-1} \frac{w_0^{(k)}(1)}{k!}(2a(z))^k.$$

On the other hand, by differentiating both sides of $a(z)w_0(z) = (z^n - 1)/2n$ $k+1$ times and substituting $z = 1$, we have

$$w_0^{(k)}(1) = \frac{(n-1)(n-2)\cdots(n-k)}{k+1},$$

which completes the proof. (Q.E.D.)

Here we define

$$y := |a(z)|^2 = \frac{1-\cos\xi}{2}. \qquad (\xi = \arg z)$$

**Lemma 3.6.** *For any $k \in \mathbf{N}$, the following relations hold:*

$$a(z)^{2k-1} + a(\bar{z})^{2k-1} = y^k \lambda_{2k-1}(y)$$
$$a(z)^{2k} + a(\bar{z})^{2k} = y^k \lambda_{2k}(y),$$

*where*

$$\begin{pmatrix} \lambda_{2k-1}(y) \\ \lambda_{2k}(y) \end{pmatrix} = 2\begin{pmatrix} -1 & -2 \\ 2y & 4y-1 \end{pmatrix}^k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

**Lemma 3.7.** If the symbol vector $\vec{m}(z)$ satisfies the moment condition of order $N$, then it holds that

$$|m_0(z)|^2 \equiv 1 \pmod{y^{N+1}}.$$

*Proof:* Since $\vec{m}(z) = A(z^n)\vec{b}(z)$ and $\sqrt{n}A(z^n)$ is unitary, we have

$$\left|\vec{m}(z)\right| = \left|n^{-1/2}\vec{b}(z)\right| = 1,$$

which leads the lemma since $m_j(z) \equiv 0 \left(\mathrm{mod}\ a(z)^{N+1}\right)$ $(j = 1, \ldots, n-1)$. (Q.E.D.)

**Lemma 3.8.** *It holds that*[2)]

$$|w_0(z)|^2 = \prod_{k=1}^{n-1}\left(1 - \frac{2y}{1-\cos(2\pi k/n)}\right).$$

Here we define $d_k(y)$ for $k \in \mathbf{N}$ and $D(a)$ as follows:

$$d_k(y) := \frac{1}{|w_0(z)|^{2(k+1)}} = 1 + \sum_{j\in\mathbf{N}} d_{k,j}\, y^j$$
$$D(a) := \frac{1}{w_0(z)} = 1 + \sum_{j\in\mathbf{N}} D_j\, a(z)^j.$$

We can see $d_{k,j} > 0$ for all $j, k \in \mathbf{N}$ from Lemma 3.8.

For any (formal) series $F = \sum_{j \in \mathbf{N} \cup \{0\}} F_j$, we put

$$[F]_N := \sum_{j=0}^{N} F_j.$$

**Lemma 3.9.** *The following relations hold:*

$$g_1(z) = 1 - (n-1)a(z),$$
$$g_k(z) \equiv [D(a)]_k g_{k-1}(z) + 2n\eta_{k-2} a(z)^{k-1} [D(a)]_1 \pmod{a(z)^{k+1}}. \qquad (k = 2, \ldots, N)$$

*Proof:* First, from eq. (3.5) we have

$$g_1(z) = D(a)\left(1 + 2na(z)^2 u_0(z)\right)$$
$$\equiv [D(a)]_1 \pmod{a(z)^2}.$$

Noting that $\deg g_1(z) = 1$, we obtain from Lemma 3.5

$$g_1(z) = [D(a)]_1 = 1 - (n-1)a(z).$$

Next, from eq. (3.5) we have

$$g_k(z) = D(a)\left(g_{k-1}(z) + 2n\eta_{k-2} a(z)^{k-1} + (2n)^k a(z)^{k+1} u_{k-1}(z)\right)$$
$$\equiv \left(1 + D_1(z)a(z) + \cdots + D_k(a)a(z)^k\right) g_{k-1}(z) + 2n\eta_{k-2}(1 + D_1(a)a(z))a(z)^{k-1} \left(\operatorname{mod} a(z)^{k+1}\right)$$

which leads the lemma. (Q.E.D.)

**Lemma 3.10.** *If a symbol vector $\vec{m}(z)$ satisfies the moment condition of order N, then it holds that*

$$\left| g_N(z) + (2na(z))^N \eta_{N-1} \right|^2 = [d_N(y)]_N.$$

*Proof:* From eq. (3.5) we get

$$m_0(z) = w_0(z)^{N+1}\left(g_N(z) + (2na(z))^N \eta_{N-1}\right).$$

And from Lemma 3.7 we have

$$\left| g_N(z) + (2na(z))^N \eta_{N-1} \right|^2 = |m_0(z)|^2 d_N(y)$$
$$\equiv [d_N(y)]_N \pmod{y^{N+1}}.$$

On the other hand, since $g_N(z) + (2na(z))^N \eta_{N-1}$ is a polynomial of $a(z)$ of degree $N$, $\left| g_N(z) + (2na(z))^N \eta_{N-1} \right|^2$ becomes a polynomial of $y$ of degree $N$. So the lemma holds. (Q.E.D.)

**Lemma 3.11.** *There exists a polynomial $f(x)$ with real coefficients such that*

$$[d_N(y)]_N = |f(a(z))|^2.$$

*Proof:* Let us factor

$$[d_N(y)]_N = \prod_k (1+s_k y) \prod_k (1+t_k y+u_k y^2),$$

where $s_k, t_k, u_k \in \mathbf{R}$ and any $1+t_k y+u_k y^2$ has no zero point in $\mathbf{R}$.

We first note that each $s_k>0$ because of the positivity of all coefficients of $[d_N(y)]_N$. On the other hand, for a positive $s$, we can find $\sigma \in \mathbf{R}$ such that $1+sy=|1+\sigma a(z)|^2$. In fact, it suffices to take $\sigma = 1 \pm \sqrt{1+s}$.

Next we note that each $t_k, u_k$ satisfies that $t_k^2 - 4u_k < 0$. On the other hand, for $t, u \in \mathbf{R}$ which satisfy $t^2 - 4u < 0$, we can find $\tau, \rho \in \mathbf{R}$ such that

$$\begin{aligned} 1+ty+uy^2 &= \left|1+\tau a(z)+\rho a(z)^2\right|^2 \\ &= 1+(\tau^2-2\tau-2\rho)y+(\rho^2-2\rho\tau+4\rho)y^2. \end{aligned}$$

In fact, let

$$\begin{aligned} \rho(\tau) &= \frac{1}{2}(\tau^2-2\tau-t) \\ g(\tau) &:= \rho(\tau)^2-2\tau\rho(\tau)+4\rho(\tau), \end{aligned}$$

then $g(2)=t^2/4$, which implies that $g(\tau)=u$ has at least one real root since $g(\tau) \longrightarrow +\infty$ as $|\tau| \longrightarrow \infty$.

So we have $f(x)$ of the lemma from the above discussion. (Q.E.D.)

Hence we have from Lemmas 3.10 and 3.11

$$g_N(z)+(2na(z))^N \eta_{N-1} = f(a(z)),$$

and we can determine $\eta_0, \ldots, \eta_{N-1}$. Also $u_0(z), \ldots, u_{N-1}(z)$ are determined from Lemma 3.9 and eq. (3.5). Moreover, we have $R\vec{q}, RS\vec{q}, \ldots, RS^{N-1}\vec{q}$ from Lemma 3.4. This means that $P\vec{q}, PS\vec{q}, \ldots, PS^{N-1}\vec{q}$ are determined.

**Theorem 3.1.** *We can determine $\vec{p}_1, \ldots, \vec{p}_N$ when $P\vec{q}, PS\vec{q}, \ldots, PS^{N-1}\vec{q}$ are given.*

*Proof:* Any vector in $\{P\vec{q}, PS\vec{q}, \ldots, PS^{N-1}\vec{q}\}$ is not zero vector since $u_{k-1}(z) \neq 0$ $(k=1, \ldots, N)$. It holds that

$$PS^{N-1}\vec{q} = q_1 s_{2,1} s_{3,2} \cdots s_{N,N-1} \vec{p}_N,$$

where $s_{j,k}$ is the $j-k$ component of the matrix $S$. So we obtain

$$\begin{aligned} \vec{p}_N &= PS^{N-1}\vec{q} \Big/ \left|PS^{N-1}\vec{q}\right| \\ S^{N-1}\vec{q} &= {}^t\left(0, \ldots, 0, \left|PS^{N-1}\vec{q}\right|\right), \end{aligned}$$

where the sign of $\vec{p}_N$ is arbitrary since we need only $\vec{p}_N {}^t\vec{p}_N$ for the symbol vector $\vec{m}(z)$.

Here we use the inductive method. Assume that $\vec{p}_N, \ldots, \vec{p}_{N-m}$ and $S^{N-1}\vec{q}, \ldots, S^{N-m-1}\vec{q}$ are given. Let us define $(\vec{x})_k$ for a vector $\vec{x}$ as the $k$-th component of $\vec{x}$.

It holds that

$$PS^{N-m-2}\vec{q} = (\vec{p}_1, \ldots, \vec{p}_N)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ (S^{N-m-2}\vec{q})_{N-m-1} \\ \vdots \\ (S^{N-m-2}\vec{q})_N \end{pmatrix}$$

$$= \sum_{k=N-m-1}^{N} (S^{N-m-2}\vec{q})_k \vec{p}_k. \tag{3.6}$$

Recalling that $\vec{p}_N, \ldots, \vec{p}_{N-m}$ are unit vectors, we have for $j = N-m, \ldots, N$

$$\begin{aligned} \langle PS^{N-m-2}\vec{q}, \vec{p}_j \rangle &= \sum_{k=N-m-1}^{N} (S^{N-m-2}\vec{q})_k \langle \vec{p}_k, \vec{p}_j \rangle \\ &= \sum_{k=N-m-1}^{j-1} (S^{N-m-2}\vec{q})_k s_{j,k} + (S^{N-m-2}\vec{q})_j + \sum_{k=j+1}^{N} (S^{N-m-2}\vec{q})_k s_{k,j}. \end{aligned} \tag{3.7}$$

Noticing that it holds that

$$\sum_{k=N-m-1}^{j-1} (S^{N-m-2}\vec{q})_k s_{j,k} = (s_{j,1}, \ldots, s_{j,j-1}, 0, \ldots, 0)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ (S^{N-m-2}\vec{q})_{N-m-1} \\ \vdots \\ (S^{N-m-2}\vec{q})_N \end{pmatrix}$$

$$= (S^{N-m-1}\vec{q})_j,$$

we can rewrite eq. (3.7) as follows:

$$\begin{aligned} \langle PS^{N-m-2}\vec{q}, \vec{p}_N \rangle &= (S^{N-m-1}\vec{q})_N + (S^{N-m-2}\vec{q})_N \\ \langle PS^{N-m-2}\vec{q}, \vec{p}_{N-1} \rangle &= (S^{N-m-1}\vec{q})_{N-1} + (S^{N-m-2}\vec{q})_{N-1} + (S^{N-m-2}\vec{q})_N s_{N,N-1} \\ &\vdots \\ \langle PS^{N-m-2}\vec{q}, \vec{p}_{N-m} \rangle &= (S^{N-m-1}\vec{q})_{N-m} + (S^{N-m-2}\vec{q})_{N-m} + \sum_{k=N-m+1}^{N} (S^{N-m-2}\vec{q})_k s_{k,N-m}. \end{aligned}$$

So we get $(S^{N-m-2}\vec{q})_N, \ldots, (S^{N-m-2}\vec{q})_{N-m}$ from the assumption.

From eq. (3.6), we know that

$$\begin{aligned}\vec{\zeta}_{N-m-1} &:= (S^{N-m-2}\vec{q})_{N-m-1}\vec{p}_{N-m-1} \\ &= PS^{N-m-2}\vec{q} - \sum_{k=N-m}^{N}(S^{N-m-2}\vec{q})_k \vec{p}_k\end{aligned}$$

is known. Also we know $\vec{\zeta}_{N-m-1} \neq \vec{0}$ since

$$\begin{aligned}PS^{N-1}\vec{q} \neq \vec{0} &\Longrightarrow (S^{N-m-2}\vec{q})_{N-m-1} \neq 0 \\ &\Longrightarrow \vec{p}_{N-m-1} \neq \vec{0}.\end{aligned}$$

So we get

$$\vec{p}_{N-m-1} = \vec{\zeta}_{N-m-1} \Big/ \left|\vec{\zeta}_{N-m-1}\right|$$

$$(S^{N-m-2}\vec{q})_{N-m-1} = \left|\vec{\zeta}_{N-m-1}\right|.$$

From the above argument, we obtain

$$S^{N-m-2}\vec{q} = {}^t\Big(0, \ldots, 0, (S^{N-m-2}\vec{q})_{N-m-1}, \ldots, (S^{N-m-2}\vec{q})_N\Big).$$

(Q.E.D.)

Hence $\vec{p}_1, \ldots, \vec{p}_N$ are determined and we have the symbol vector $\vec{m}(z)$ from eq. (3.3). It should be noted that $\vec{p}_1, \ldots, \vec{p}_N$ are not determined as a set of unit vectors.

The scaling function given above is called the scaling function with minimal length. The scaling function with minimal length has the smallest support in the functions which satisfy moment condition of order $N$.

## 4. Wavelet with Zero Point Condition

If the scaling symbol $m_0(z)$ has a zero point, then the scaling and wavelet functions are expected to become smoother under the same moment condition. In this section, we try to make the scaling symbol $m_0(z)$ having a zero point. For this purpose, we will have to take more unit vectors.

Assume that the scaling symbol $m_0(z)$ satisfies the moment condition of order $N$ and there exists $z_0$ $(|z_0| = 1)$ such that $m_0(z_0) = 0$. Then we have

$$|m_0(z)|^2 \equiv 0 \quad \left(\text{mod } (1 - \frac{y}{y_0})^2\right), \tag{4.1}$$

where $y_0 := |a(z_0)|^2$.

Here we take a family of real unit vectors $\{\vec{p}_1, \ldots, \vec{p}_l\}$ for $l > N$, then we have from Lemma 3.10

$$|m_0(z)|^2 = |w_0(z)|^{2(N+1)}\left\{[d_N(y)]_N + y^{N+1}\sum_{k=N}^{l-1} s_k(y)\Big(|w_0(z)|^2 y\Big)^{k-N}\right\}, \tag{4.2}$$

where each $s_k(y)$ $(k=N,\dots,l-1)$ is denoted as

$$s_k(y)=\sum_{j=0}^{n-1}\sigma_{k,j}y^j, \qquad (k=N,\dots,l-1)$$

and the coefficients of $s_k(y)$ satisfy the following relations:[2)]

$$\text{Case where } n=2;\ \ \sigma_{k,0}+\frac{\sigma_{k,1}}{2}=0$$

$$\text{Case where } n=3;\ \ \sigma_{k,0}+\frac{\sigma_{k,1}}{2}+\frac{3\sigma_{k,2}}{8}=0$$

$$\vdots$$

So if we can make $m_0(z)$ which satisfies eqs. (4.1) and (4.2), then we have a symbol satisfying the moment condition of order $N$ and having a zero point.

## 5. Examples

In this section we investigate how the smoothness of wavelet system varies along with the zero point of $m_0(z)$ through some examples. As a result, we see that there exists the cases where the wavelet system does not become smooth.

Case where $n=2$, $N=1$

Assume first $n=2$, $N=1$ and $l=2$. Then from eq. (4.2) we have

$$|m_0(z)|^2=\left|w_0(z)^2\right|^2\left\{1+2y+y^2(\sigma-2\sigma y)\right\},$$

where $|w_0(z)|^2=1-y$, and from eq. (4.1) we have

$$|m_0(z)|^2\Big|_{y=y_0}=0, \qquad \frac{d}{dy}\Big|_{y=y_0}|m_0(z)|^2=0.$$

So we obtain

$$|m_0(z)|^2=\left|w_0(z)^2\right|^2\left(1-\frac{y}{y_0}\right)^2\left(1+2(2\pm\sqrt{5}\,)y\right)$$

$$y_0=\frac{-1\pm\sqrt{5}}{4}.$$

In this case, we cannot choose $y_0$ arbitrarily. That is, the zero point is fixed.

On the other hand, it holds that

$$m_0(z)=w_0(z)^2\left\{g_1(z)+4\eta_0 a(z)+\left(4a(z)\right)^2\left(\eta_1 w_0(z)+a(z)u_1\right)\right\},$$

where

$$g_1(z)=1-a(z), \qquad u_1=\langle R S\vec{q}, \vec{\gamma}(z)\rangle=RS\vec{q}.$$

From this, we obtain

$$\eta_0=\frac{4+\sqrt{5+2\sqrt{5}}}{4},\quad \eta_1=\frac{3+\sqrt{5}+2\sqrt{5+2\sqrt{5}}}{16},\quad R S\vec{q}=\frac{-2+(\sqrt{5}-1)\sqrt{5+2\sqrt{5}}}{16},$$

and $R\vec{q}=-1/4$ from eq. (3.5) and Lemma 3.9. Hence we have the symbol vector $\vec{m}(z)$ from Theorem 3.1.

Next, assume $n=2, N=1$ and $l=3$. In this case, $|m_0(z)|^2$ is parametrized by $y_0$. We have

$$|m_0(z)|^2=\left|w_0(z)^2\right|^2\left\{1+2y+y^2(s_1(y)+s_2(y)|w_0(z)|^2y)\right\}, \tag{5.1}$$

where

$$s_k(y)=\sigma_k-2\sigma_k y. \qquad (k=1,2)$$

From eq. (4.1) we have

$$\sigma_1=\frac{16y_0^3-8y_0^2-8y_0+3}{y_0^2(2y_0-1)^3}, \qquad \sigma_2=\frac{8y_0^2+4y_0-2}{y_0^3(2y_0-1)^3}.$$

Thus, we can obtain the wavelet system parametrized by $y_0$ $(0<y_0\leq 1)$. It should be noted that there exists a range of $y_0$ where the right hand side of eq. (5.1) is not always positive.

Figure 1 shows the scaling and wavelet functions when $n=2, N=1, l=2$ and fixed $y_0=(-1+\sqrt{5})/4\simeq 0.309017$. Figure 2 shows the scaling and wavelet functions when $n=2, N=1, l=3$ and $y_0=0.8$.

Case where $n=2$, $N=2$

Assume $n=2$, $N=2$ and $l=3$. This time, the degree of freedom of the zero point $y_0$ does not remain. Through a similar process of the above, we have

$$|m_0(z)|^2=\left|w_0(z)^3\right|^2\left(1-\frac{y}{y_0}\right)^2(1+8.89221y+49.7152y^2),$$

where $|w_0(z)|^2=1-y$ and $y_0=-\frac{1}{6}+\frac{1}{3}\left(\frac{7}{2}\right)^{1/3}\simeq 0.339431$. Therefore we have

$$\eta_0=2.7829, \qquad \eta_1=2.51207, \qquad \eta_2=0.761169,$$

$$R\vec{q}=-0.25, \qquad RS\vec{q}=-0.570725, \qquad RS^2\vec{q}=-0.123513.$$

So we can obtain the scaling and wavelet functions.

Next, assume $n=2$, $N=2$ and $l=4$. In this case, $|m_0(z)|^2$ is parametrized by $y_0$. We have

$$|m_0(z)|^2=\left|w_0(z)^3\right|^2\left\{1+3y+6y^2+y^3\left(s_2(y)+s_3(y)|w_0(z)|^2y\right)\right\},$$

where

$$s_2(y)=\frac{2(2-3y_0-6y_0^2-12y_0^3+24y_0^4)}{y_0^3(-1+2y_0)^3}(1-2y)$$

$$s_3(y)=\frac{-3+2y_0+12y_0^2+24y_0^3}{y_0^4(-1+2y_0)^3}(1-2y).$$

Thus, we can obtain the wavelet system parametrized by $y_0$.

Figure 3 shows the scaling and wavelet functions when $n=2$, $N=1$, $l=2$ and fixed $y_0=-\frac{1}{6}+\frac{1}{3}\left(\frac{7}{2}\right)^{1/3}\simeq 0.339431$. Figure 4 shows the scaling and wavelet functions when $n=2$, $N=1$, $l=3$ and $y_0=0.8$.

Case where $n=3$, $N=1$

Assume $n=3$, $N=1$ and $l=2$. Then we have

$$|m_0(z)|^2=\left|w_0(z)^2\right|^2\left\{1+\frac{16}{3}y+y^2(\sigma_0+\sigma_1 y+\sigma_2 y^2)\right\},$$

where

$$|w_0(z)|^2=1-\frac{8}{3}y+\frac{16}{9}y^2,$$

$$\sigma_0=\frac{-9-16y_0+64y_0^2}{y_0^2(3-8y_0+8y_0^2)},\quad \sigma_1=-\frac{2(-3-8y_0+16y_0^2+64y_0^3)}{y_0^3(3-8y_0+8y_0^2)},\quad \sigma_2=\frac{8(-3+y_0+32y_0^2)}{3y_0^3(3-8y_0+8y_0^2)}.$$

Therefore we know that the wavelet system is decided according to the choice of $y_0$.

Figure 5 shows the scaling and wavelet functions when $n=3$, $N=1$, $l=1$ and $y_0=0.4$. Figure 6 shows the scaling and wavelet functions when $n=3$, $N=1$, $l=1$ and $y_0=0.8$.

Case where $n=3$, $N=2$

Assume $n=3$, $N=1$ and $l=3$. Then we have

$$|m_0(z)|^2=\left|w_0(z)^3\right|^2\left\{1+8y+\frac{112}{3}y^2+y^3(\sigma_0+\sigma_1 y+\sigma_2 y^2)\right\},$$

where

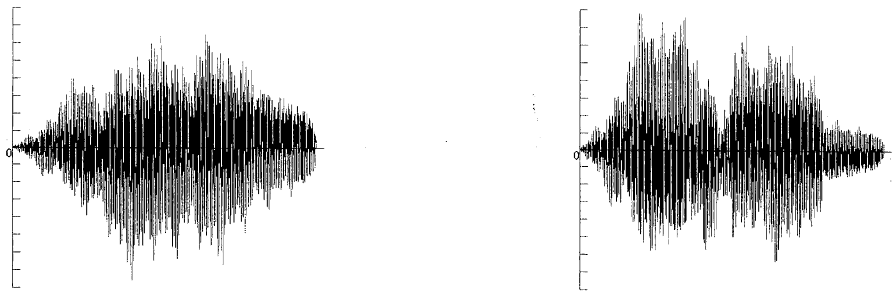
$$\sigma_0=\frac{4(-3-13y_0-24y_0^2+112y_0^3)}{y_0^3(3-8y_0+8y_0^2)},\quad \sigma_1=\frac{9+48y_0+72y_0^2-256y_0^3-896y_0^4}{y_0^4(3-8y_0+8y_0^2)},$$

$$\sigma_2=\frac{4(-9-24y_0+32y_0^2+448y_0^3)}{3y_0^4(3-8y_0+8y_0^2)}.$$

Therefore we know that the wavelet system is parametrized by $y_0$ $(0<y_0\leq 1)$.

Figure 7 shows the scaling and wavelet functions when $n=3$, $N=2$, $l=3$ and $y_0=0.4$. Figure 8 shows the scaling and wavelet functions when $n=3$, $N=2$, $l=3$ and $y_0=0.8$.

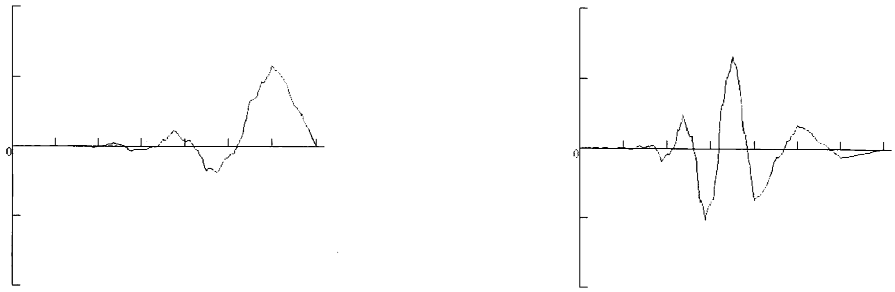
$n = 2,\ N = 1,\ l = 2,\ y_0 = (-1 + \sqrt{5})/4$(fixed)



(a)Scaling function (b)Wavelet function

**Fig. 1.** The scaling and wavelet functions.

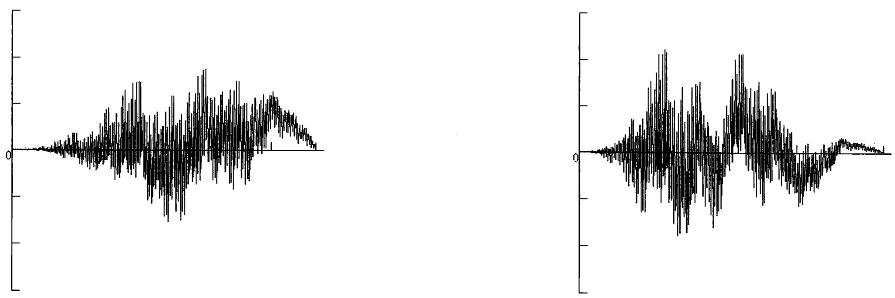
$n = 2,\ N = 1,\ l = 3,\ y_0 = 0.8$



(a)Scaling function (b)Wavelet function

**Fig. 2.** The scaling and wavelet functions.

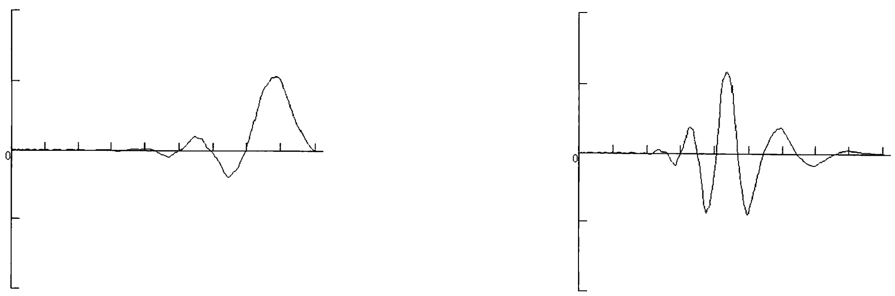
$n = 2,\ N = 2,\ l = 3,\ y_0 = 0.339431$(fixed)



(a)Scaling function (b)Wavelet function

**Fig. 3.** The scaling and wavelet functions.

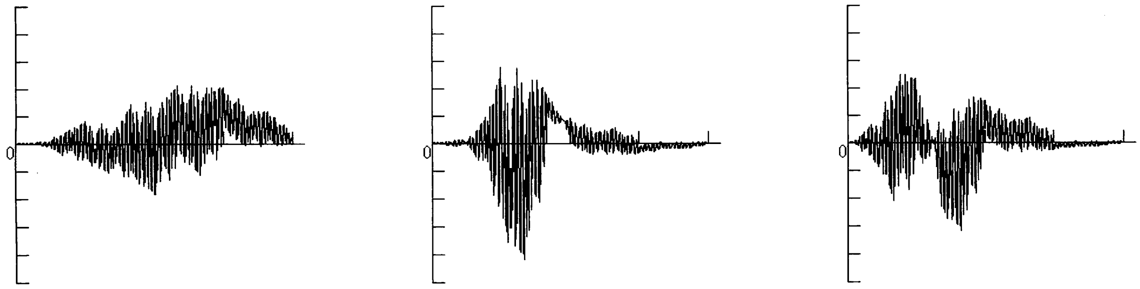
$n = 2,\ N = 2,\ l = 4,\ y_0 = 0.8$



(a)Scaling function (b)Wavelet function

**Fig. 4.** The scaling and wavelet functions.

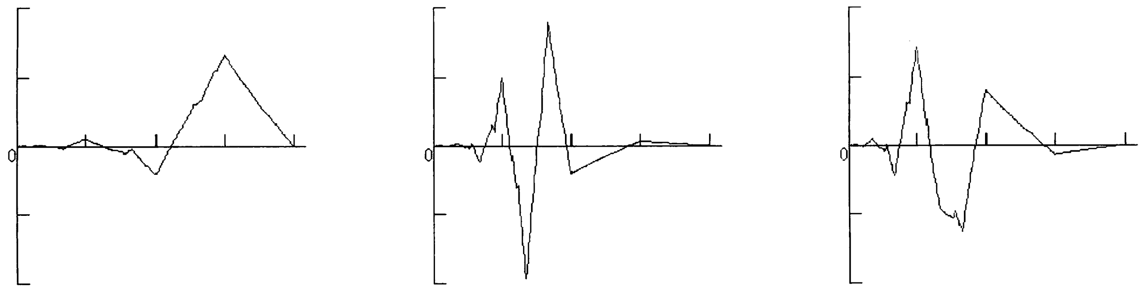
$n=3,\ N=1,\ l=2,\ y_0=0.4$



(a)Scaling function (b)Wavelet function (c)Wavelet function

**Fig. 5.** The scaling and wavelet functions.

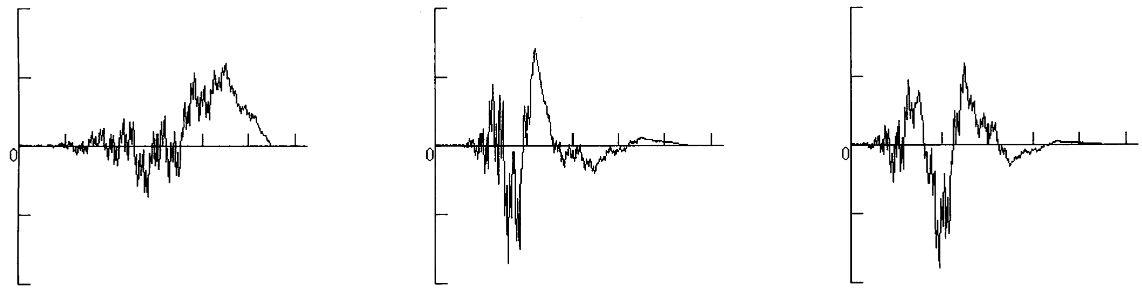
$n=3,\ N=1,\ l=2,\ y_0=0.8$



(a)Scaling function (b)Wavelet function (c)Wavelet function

**Fig. 6.** The scaling and wavelet functions.

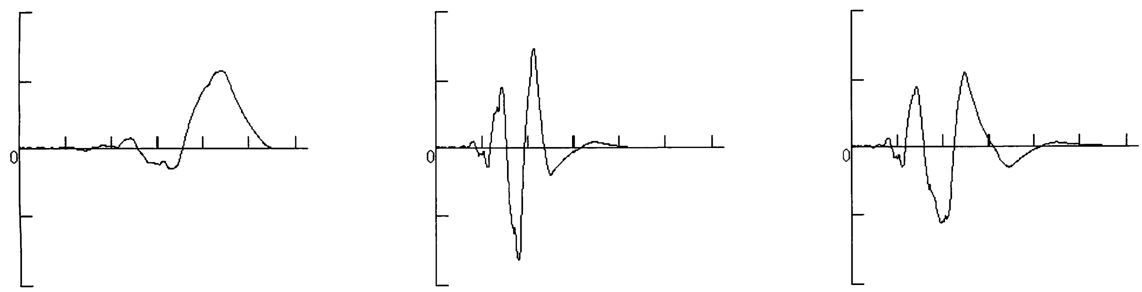
$n=3,\ N=2,\ l=3,\ y_0=0.4$



(a)Scaling function (b)Wavelet function (c)Wavelet function

**Fig. 7.** The scaling and wavelet functions.

$n=3,\ N=2,\ l=3,\ y_0=0.8$



(a)Scaling function (b)Wavelet function (c)Wavelet function

**Fig. 8.** The scaling and wavelet functions.

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