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The Homeomorphism Groups of Noncompact 2-Manifolds

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Abstract

Suppose M is a noncompact connected 2-manifold and let $\mathscr{H}(M)$ denote the homeomorphism group of M with the compact-open topology. In this paper we determine the condition on the end of M under which $\mathscr{H}(M)$ is an ℓ^2-manifold. It is shown that $\mathscr{H}(M)$ is an ℓ^2-manifold iff (i) $M = N \setminus (F \cup A)$, where N is a compact connected 2-manifold, F is a finite subset of Int N and A is a 0-dimensional compact subset of ∂N and (ii) the group $\mathscr{H}_+(A)$ of order preserving homeomorphisms of A is discrete.

Key Words : *2-manifolds; Ends; Homeomorphism groups; ℓ^2-manifolds.*

1. Introduction

In this paper we will investigate the homeomorphism groups of noncompact 2-manifolds from the viewpoint of infinite-dimensional manifolds. Suppose M^n is a topological n-manifold and let $\mathscr{H}(M)$ denote the homeomorphism group of M onto itself with the compact-open topology (the topology of uniform convergence on every compact subset). R. D. Anderson[1)] showed that $\mathscr{H}(\mathbb{R})$, the group of orientation-preserving homeomorphisms of the real line $\mathbb{R}$, is homeomorphic to ℓ^2 (the Hilbert space of square summarable real sequences). After this result, it was conjectured that the homeomorphism group $\mathscr{H}(M)$ is an ℓ^2-manifold for any compact n-manifold and some noncompact n-manifolds M^n.

The development of the theory of infinite-dimensional topological manifold reached its climax in the characterization theorems of various infinite-dimensional topological manifolds. In particular, a separable, completely metrizable space X is an ℓ^2-manifold iff X is an ANR (absolute neighborhood retract) and ℓ^2-stable ($X \times \ell^2 \cong X$).[11)] Since $\mathscr{H}(M)$ is always ℓ^2-stable,[3)] the conjecture is reduced to the problem of whether $\mathscr{H}(M)$ is an ANR. It is known that $\mathscr{H}(M)$ is always locally contractible when either M is compact or M is the interior of a compact manifold.[2)] However, the infinite-dimensionality of $\mathscr{H}(M)$ has prevented the detection of ANR property. For this reason the conjecture is known to be true only for $n = 1$ and 2, and still remains open for $n \geq 3$. In the one-dimensional case, $\mathscr{H}(M)$ is an ℓ^2-manifold even if M is noncompact.[1)] In the two dimensional case, the conjecture has been proved for any compact 2-manifolds.[6),7)] On the other hand, if M is a noncompact 2-manifold, then in general $\mathscr{H}(M)$ is not necessarily even locally path connected. (Consider the case where M has infinitely many handles, cf. Ref. 10) Example 5.6.1.) The purpose of this paper is to determine the condition on the end of a noncompact 2-manifold M under which $\mathscr{H}(M)$ is an ℓ^2-manifold.

Suppose M is a noncompact connected PL 2-manifold and X is a compact subpolyhedron of M. (Note that every 2-manifold admits a PL-triangulation.) Let $\mathscr{H}_X(M)$ (respectively $\mathscr{H}_{X\cup\partial}(M)$) denote the subgroup of $\mathscr{H}(M)$ consisting of the homeomorphisms of M which are identity on X (respectively $X\cup\partial M$). Consider the following condition on M:

$(*)$ $M=N\setminus(F\cup A)$, where N is a compact connected 2-manifold, F is a finite subset of Int N and A is a 0-dimensional compact subset of ∂N.

Proposition 1. *Suppose M is a noncompact connected PL 2-manifold and X is a compact subpolyhedron of M. Then the following conditions are equivalent:*
(i) *$\mathscr{H}_{X\cup\partial}(M)$ is an ℓ^2-manifold.*
(ii) *$\mathscr{H}_{X\cup\partial}(M)$ is locally path connected.*
(iii) *M takes the form of $(*)$.*

To ensure that the whole group $\mathscr{H}_X(M)$ is an ℓ^2-manifold, we need a further restriction on the end of M. When M has the form $(*)$, we can define the group $\mathscr{H}_+(A)$ of order preserving homeomorphisms of A as follows: Choose a finite collection of disjoint oriented arcs $\mathscr{J}=\{I_i\}_i$ in ∂N with $A\subset\cup_i I_i$ and set $A_i=A\cap I_i$. The orientation on I_i induces a linear order on A_i. Let $\mathscr{H}_+(A)\equiv\mathscr{H}_+(A;\mathscr{J})=\{f\in\mathscr{H}(A): f(A_i)=A_i$ and $f|A_i$ is order preserving for each $i\}$, equipped with the compact open topology. The following is the main result of this paper.

Theorem 1. *Suppose M is a noncompact connected PL 2-manifold and X is a compact subpolyhedron of M. Then the following conditions are equivalent:*
(i) *$\mathscr{H}_X(M)$ is an ℓ^2-manifold.*
(ii) *$\mathscr{H}_X(M)$ is locally path connected.*
(iii) *M takes the form $(*)$ and $\mathscr{H}_+(A)$ is discrete.*

Theorem 1 implies that local path connectedness is the unique obstruction. The similar problem for PL-homeomorphisms and diffeomorphisms will be investigated in a subsequent paper.

2. Basic Facts on ℓ^2-Manifolds and Homeomorphism Groups

Throughout the paper we assume that all spaces are separable and metrizable and that all maps are continuous. When A is a subset of a space X, $\mathrm{Fr}_X A$, $\mathrm{Int}_X A$ and $cl_X A$ denote the topological frontier, interior and closure of A in X. On the other hand, ∂X and Int M denote the boundary and interior of a manifold M. The notation $\cong$ means "homeomorphic", while $\simeq$ means "homotopic"

A space X is said to be an ℓ^2-manifold if each point has an open neighborhood which is homeomorphic to ℓ^2. H. Toruńczyk[11] showed that ℓ^2-manifolds are characterized by their ANR-property and ℓ^2-stability. A space X is said to be an ANR (*absolute neighborhood* retract) if for every map $f: B \to X$ from a closed subset B of a space Y there exist a neighborhood U of B in Y and an extension $\tilde{f}: U \to X$ of f. If, in addition, we can always take $U = Y$, then X is said to be an AR.

Fact 2.1. *A* space X is an ANR iff each point of X has a neighborhood which is an ANR. In particular, a topological group is an ANR iff the unit element has an ANR neighborhood.[4]

A space X is said to be ℓ^2-stable if $X \times \ell^2 \cong X$.

Fact 2.2. *A* space X is an ℓ^2-manifold iff X is a separable, completely metrizable ANR and is ℓ^2-stable.[11]

Suppose X is a space. By $\mathscr{H}(X)$ we denote the group of homeomorphisms of X onto itself with the compact open topology. When A, C are subsets of X, we set $\mathscr{H}_C(X, A) = \{f \in \mathscr{H}(X) \mid f(A) = A$ and $f|_C = id_C\}$. Suppose M is an n-manifold ($m \geq 1$) and A is a proper closed subset of M. Then (i) $\mathscr{H}_A(M)$ is a separable, completely metrizable topological group[12] and (ii) $\mathscr{H}_A(M)$ is ℓ^2-stable.[3]. Therefore, the next assertion follows from Facts 2.1 and 2.2.

Fact 2.3. Suppose M is an n-manifold ($n \geq 1$) and A is a proper closed subset of M. Then $\mathscr{H}_A(M)$ is an ℓ^2-manifold iff the identity id_M has an ANR neighborhood in $\mathscr{H}_A(M)$.

Luke-Mason[6),7)] verified the ANR property in the 2-dimensional compact case.

Fact 2.4. (i) If M is a compact 2-manifold, then $\mathscr{H}(M)$ is an ANR.[6]
(ii) If M is a compact PL 2-manifold and X is a compact subpolyhedron, then $\mathscr{H}_X(M)$ is an ANR.[cf.5]

In Ref. 5) it is shown that $\mathscr{H}(X)$ is an ANR for every compact 2-dimensional polyhedron. This follows from Fact 2.4.(i) by a cutting argument. Fact 2.4. (ii) (the relative version) also follows from Fact 2.4. (i) by a similar cutting argument.

Finally, we list a fact on end compactifications.

Fact 2.5. Suppose X is a locally connected compact metric space, A is a 0-dimensional compact subset of X and C is a compact subset of $X \setminus A$. If (i) $X \setminus A$ is dense in X and (ii) $U \setminus A$ is connected for any connected open subset U of X (for example, X is a compact 2-manifold), then the restriction map $\mathscr{H}_C(X, A) \to \mathscr{H}_C(X \setminus A): h \mapsto h|_{X \setminus A}$ is a homeomorphism.

3. Local Path Connectedness of $\mathscr{H}(M)$ and the Ends of M

Suppose M is a noncompact connected PL 2-manifold and X is a compact subpolyhedron of M. In this section we will show that if $\mathscr{H}_X(M)$ or $\mathscr{H}_{X \cup \partial}(M)$ is locally path connected, then M takes the form (*) :

(∗) $M = N \setminus (F \cup A)$, where N is a compact connected 2-manifold, F is a finite subset of Int N and A is a 0-dimensional compact subset of ∂N.

Lemma 3.1. *M takes the form (∗) iff there exists a (nonempty) compact connected 2-submanifold N_0 of M such that each component L of $cl(M \setminus N_0)$* satisfies the following condition:
(∗∗) $L = \bigcup_{i=1}^{\infty} L_i$, *where*
(i) *each L_i is a compact connected 2-submanifold of L,* $\mathrm{Fr}_M L \subset \mathrm{Int}_L L_1$ and $L_i \subset \mathrm{Int}_L L_{i+1}$,
(ii) (a) *each component K of $cl(L \setminus L_i)$ is noncompact,* (b) *$K \cap L_{i+1}$ is connected,* (c) *$K \cap L_i$ is an arc or a circle, and*
(iii) *each L_i is a disk with m holes ($m = m(L)$ is independent of i).*

Proof. If L satisfies (∗∗), then from (iii) each component H of $cl(L_{i+1} \setminus L_i)$. is either (a) a disk D such that $D \cap L_i$ is an arc or (b) an annulus A such that $A \cap L_i$ is a boundary circle. Let $\ell_i \equiv$ the number of the annulus components of $cl(L_{i+1} \setminus L_i)$. Then $\ell_i \geq \ell_{i+1}$, and we may assume that $\ell_i \equiv \ell$ (constant) by omitting finitely many L_i's. Hence L takes the form (∗), where N is a disk with holes and $\# F = \ell$. This implies that M itself takes the form (∗). □

Let $\mathscr{D}$ denote the subset of $\mathscr{H}_\partial(M)$ consisting of all Dehn twists on M. From the definition of the compact-open topology, if $\mathscr{H}_X(M)$ or $\mathscr{H}_{X \cup \partial}(M)$ is locally path connected then $\mathscr{D}$ satisfies the following condition :
(#) There exists a compact subset C of M such that $(\#)_C$: if $h \in \mathscr{D}$ and $h|_C = id$ then $h \simeq id_M$.

The following properties of Dehn twists will be used in a criterion of the condition (#) in Lemma 3.5.

Lemma 3.2. (1) *Every Dehn twist along a meridian of a handle in M is not homotopic to id_M.*
(2) *Suppose $D \subset M$ is a sphere with $n+1$ holes, $n \geq 4$, and let S and S_i ($i = 1, \cdots, n$) denote the boundary circles of D. Suppose C is a circle in D which encircles exactly two holes S_1 and S_2 and h is the Dehn twist along C. If there exists a closed set E of M such that $D \cap E \subset S$ and $D \cup E$ is a retract of M, then $h \not\simeq id_M$.*

Proof. Statement (1) is verified by a simple π_1-calculation. The next statement also follows from a π_1-calculation.
(2′) Suppose D is a sphere with n holes, $n \geq 4$, and let S_i ($i = 1, \cdots, n$) denote the boundary circles of D. Suppose C is a circle in D which encircles exactly two holes S_1 and S_2, ℓ is a circle in D which encircles exactly two holes S_2 and S_3 and h is the Dehn twist along C. Then $h\ell \not\simeq \ell$ in D.

Statement (2) reduces to (2′) by (a) retracting M onto $D \cup E$ and (b) capping S by a disk and mapping E into this disk. □

Lemma 3.3. *If M is a noncompact connected 2-manifold and C is a boundary circle of M, then C is a retract of M.*

Proof. Take a half ray α in M which starts from a point on C and goes toward the end of M. If A is a regular neighborhoood of $C \cup \alpha$, then $\mathrm{Fr}_M A \simeq \mathbb{R}$ (the real line). Hence we have a sequence of retractions $M \to A \to C \cup \alpha \to C$. □

Lemma 3.4. *Suppose M is a noncompact connected 2-manifold. If M is orientable and contains no handles, then for any compact subset C of M there exists a compact connected 2-submanifold N of M such that each component L of $cl(M \setminus N)$ is* (a) *noncompact and* (b) *$L \cap N$ is an arc or circle.*

Proof. Let N be a compact connected 2-submanifold with $C \subset N$ and set ℓ = the number of components of $cl(M \setminus N)$ and m = the number of components of $\mathrm{Fr}N$. Then $m \geq \ell$ and the condition (b) is equivalent to the condition $m = \ell$.

Suppose $m - \ell \geq 1$. There exists a component L of $cl(M \setminus N)$ which contains two components C_1 and C_2 of $\mathrm{Fr}N$. We can join these components by a proper arc α in L. Let A be a regular neighborhood of α in L. If $cl(L \setminus A)$ is connected, then we can find (i) a circle β in L which meets α at one point and (ii) a proper arc γ in N connecting the end points of α. Since M is orientable, a regular neighborhood of $\alpha \cup \beta \cup \gamma$ is a handle. This contradicts the assumption. Hence $cl(L \setminus A)$ has two components. Each component C_k is an arc or circle. If one of C_k is a circle, then $cl(L \setminus A)$ is connected. Hence, both C_1 and C_2 are arcs. Therefore if we replace N by $N \cup A$, then ℓ increases by 1, while m unchanges, so $m - \ell$ decreases by 1. By the repeated application of this procedure we can reach $m - \ell = 0$. To achieve (a), add all compact components of $cl(M \setminus N)$ to N. □

Lemma 3.5. *If $\mathscr{D}$ satisfies the condition (♯), then M takes the form of (∗).*

Proof. From the assumption there exists a compact connected 2-submanifold N of M which satisfies $(\sharp)_N$. Then by Lemma 3.2.(1) each component of $cl(M \setminus N)$ contains no handles, and thus contains at most two Möbius bands. Hence, we can enlarge N so that

(i) each component L of $cl(M \setminus N)$ is noncompact, orientable and contains no handles.

Thus, using Lemma 3.4 we can further enlarge N so that

(ii) $L \cap N$ is an arc or a circle for each component L of $cl(M \setminus N)$.

At this point N satisfies $(\sharp)_N$, (i) and (ii). We will show that each component L of $cl(M \setminus N)$ satisfies the condition (∗ ∗) in Lemma 3.1. This leads to the conclusion.

By the repeated application of Lemma 3.4 we can write each component L of $cl(M \setminus N)$ as $L = \cup_{i=1}^{\infty} L_i$, where

(iii) each L_i is a compact connected 2-submanifold of L, $L \cap N \subset \mathrm{Int}_L L_1$, and $L_i \subset \mathrm{Int}_L L_{i+1}$,

(iv) each component K of $cl(L \setminus L_i)$ is noncompact, $K \cap L_i$ is an arc or circle and $K \cap L_{i+1}$ is connected.

By (i) each L_i is a disk with m_i holes. It follows that $m_i \leq m_{i+1}$, since any component H of $cl(L_{i+1} \setminus L_i)$ is (a) a disk with holes, (b) $H \cap L_i$ is an arc or circle and (c) $\mathrm{Fr}_L H \neq H \cap L_i$. On the other hand, we have $m_i \leq 3$. In fact, if $m_i \geq 4$, we can apply Lemma 3.2.(2) to $D = L_i$ and $E = cl(M \setminus L)$. (Note that (a) $L_i \cap E = L \cap N$ is an arc or a circle, which is contained in a boundary circle S of L_i and (b) by (iv) and Lenuna 3.3 $K \cap L_i$ is a retract of K for any component K of $cl(L \setminus L_i)$, So $L_i \cup E$ is a retract of M.) This contradicts $(\#)_N$. Therefore we may assume that $m_i \equiv m$ (a constant) by omitting finitely many L_i's. This completes the proof.

□

We conclude this section with the proof of Proposition 1.

Proof of Proposition 1. If $\mathscr{H}_{X\cup\partial}(M)$ is locally path connected, then M takes the form $(*)$ by Lemma 3.5. Conversely, if M takes the form $(*)$, then from Fact 2.5 we have a homeomorphism $\mathscr{H}_{X\cup\partial}(M) \cong \mathscr{H}_{X\cup\partial}(N, F)$. Since M is an open set of N and X is a compact subpolyhedron of M with respect to a PL-triangulation on M, we can find a PL-triangulation of N with respect to which X is a subpolyhedron of N.[8] From Fact 2.4.(ii) $\mathscr{H}_{X\cup F\cup\partial}(N)$ is an ANR and it is an open neighborhood of id_N in $\mathscr{H}_{X\cup\partial}(N, F)$. Hence $\mathscr{H}_{X\cup\partial}(N, F)$ is an ℓ^2-manifold by Fact 2.3. □

4. Proof of Main Theorem

This final section contains the proof of Theorem 1. Throughout the section the subscript "$+$" in homeomorphism groups means "orientation-preserving" or "order-preserving". First we will list some properties of the end group $\mathscr{H}_+(A)$. Suppose I is an oriented arc and A is a 0-dimensional compact subset in I. The orientation on I induces a natural linear order on A. Let $\mathscr{H}_+(A)$ denote the group of order preserving homeomorphisms of A. Since $\mathscr{H}(A)$ is totally disconnected, $\mathscr{H}_+(A)$ is locally connected iff it is discrete.

Lemma 4.1. *$\mathscr{H}_+(I, A)$ is an ANR (or locally path connected) iff $\mathscr{H}_+(A)$ is discrete.*

Proof. We may assume that $I = [0, 1]$. We know that (i) $(0, 1) \setminus A$ is a disjoint union of (at most countably many) open intervals $U_n = (a_n, b_n)$, (ii) diam $U_n \to 0$ (as $n \to \infty$), and (iii) the correspondence $(a_n, b_n) \mapsto (f(a_n), f(b_n))$ induces an order-preserving permutation of U_n's. Each $f \in \mathscr{H}_+(A)$ has a canonical linear extension $\theta(f) \in \mathscr{H}_+(I, A)$. The map $\theta(f)$ is defined by $\theta(f)(ta_n + (1-t)b_n) = tf(a_n) + (1-t)f(b_n)$ $(0 \le t \le 1)$ (where $f(0) = 0$ and $f(1) = 1$). It follows that $\theta : \mathscr{H}_+(A) \to \mathscr{H}_+(I, A)$ is continuous and induces a homeomorphism φ : $\mathscr{H}_+(I, A) \cong \mathscr{H}_+(A) \times \mathscr{H}_{A,+}(I)$, $\varphi(h) = (h|_A, \theta(h|_A)^{-1}h)$. The restriction map $\mathscr{H}_{A,+}(I) \cong \prod_n \mathscr{H}_+(U_n) : f \mapsto (f|U_n)_n$ also induces a homeomorphism. Since $\mathscr{H}_+(U_n) \cong \ell^2$,[1] $\mathscr{H}_+(U_n)$ is an AR, and $\mathscr{H}_{A,+}(I)$ is also an AR.[4] The conclusion follows from these observations. □

Suppose N is a compact connected PL 2-manifold, F is a finite subset of Int N, A is a 0-dimensional compact subset of ∂N and X is a compact subpolyhedron of N with $X \cap (F \cup A) = \emptyset$. Let $\mathscr{J} = \{I_i\}$ be a finite collection of disjoint oriented (arcs in ∂N such that $A \subset \cup_i I_i$. (We allow that I_i is a single point.) The definition of the group $\mathscr{H}_+(A) \equiv \mathscr{H}_+(A; \mathscr{J})$ is given by §1. If $\mathscr{I}$ is another collection which refines $\mathscr{J}$, then $\mathscr{H}_+(A; \mathscr{I})$ is an open neighborhood of id_A in $\mathscr{H}_+(A; \mathscr{J})$. Hence the property "$\mathscr{H}_+(A; \mathscr{J})$ is discrete" does not depend on the choice of the collection $\mathscr{J}$.

Lemma 4.2. *$\mathscr{H}_X(N, F\cup A)$ is an ANR (or locally path connected) iff $\mathscr{H}_+(A; \mathscr{J})$ is discrete.*

Proof. Replacing each I_i by a subinterval (or a single point), we may assume that $\partial I_i \subset A_i$. Let $\mathscr{U} = \{f \in \mathscr{H}_{X\cup F}(N) | f(I_i) = I_i$ and $f|_{I_i} \in \mathscr{H}_+(I_i, A_i)$ for each i$\}$ and $\mathscr{F} = \prod_i \mathscr{H}_+(I_i, A_i)$, and let $\pi : \mathscr{U} \to \mathscr{F}$, $\pi(f) = (f|_{I_i})_i$ be the restriction map. It follows that (i) $\mathscr{U}$ is an open neighborhood of id_N in $\mathscr{H}_X(N, F\cup A)$. (ii) $\mathscr{H}_X(N, F\cup A)$ is an ANR iff $\mathscr{U}$ is an ANR, and (iii) $\mathscr{H}_+(A) \cong \prod_i \mathscr{H}_+(A_i)$, so from Lemma 4.1 $\mathscr{H}_+(A)$ is discrete iff $\mathscr{F}$ is an ANR.

Using a cone of I_i embedded in N, we can easily find a map $\lambda : \mathscr{F} \to \mathscr{U}$ such that $\pi\lambda = id_{\mathscr{F}}$ (i.e., $\lambda((g_i)_i)|_{I_j} = g_j$). This induces a homeomorphism $\varphi : \mathscr{U} \to \mathscr{H}_{X\cup F\cup(\cup_i I_i)}(N) \times \mathscr{F}$, $\varphi(f) = ((\lambda\pi(f))^{-1}f,\ \pi(f))$. The inverse is given by $\varphi^{-1}(h, (g_i)_i) = \lambda((g_i)_i)h$. Since $\mathscr{H}_{X\cup F\cup(\cup_i I_i)}(N)$ is an ANR by Fact 2.4.(ii), it follows that $\mathscr{U}$ is an ANR iff $\mathscr{F}$ is an ANR. This implies the conclusion.

The same argument applies to the "locally path connected" case (replace "ANR" by "locally path connected"). □

We can now complete the proof of Theorem 1.

Proof of Theorem 1. If M takes the form (∗), then $\mathscr{H}_X(M) \cong \mathscr{H}_X(N, F\cup A)$ by Fact 2.5. If $\mathscr{H}_X(M)$ is locally path connected, then M takes the form (∗) from Lemma 3.5 and $\mathscr{H}_+(A; \mathscr{J})$ is discrete from Lemma 4.2. Conversely, suppose M takes the form (∗) and $\mathscr{H}_+(A; \mathscr{J})$ is discrete. Since X is a compact subpolyhedron of M, X is also a compact subpolyhedron of N with respect to a PL-triangulation of N.[8] Hence $\mathscr{H}_X(N, F\cup A)$ is an ANR from Lemma 4.2, and $\mathscr{H}_X(M)$ is an ℓ^2-manifold by Fact 2.3. □

Finally, we will give some examples of 0-dimensional compact subsets A in $I = [0,1]$ for which the groups $\mathscr{H}_+(A)$ are not discrete. Note that these examples can be realized as ends of noncompact 2-manifolds.

Example. (1) (i) Let $A = \{0\} \cup \{\frac{1}{n} : n \geq 2\} \cup \{1 - \frac{1}{n} : n \geq 2\} \cup \{1\}$, and for each $n \geq 1$ take a compact subset X_n in $[\frac{1}{n+1}, \frac{1}{n}]$ such that $([\frac{1}{n+1}, \frac{1}{n}], X_n) \cong (I, A)$. If $X = \{0\} \cup (\cup_n X_n)$, then $\mathscr{H}_+(X) \cong \prod_{n=1}^{\infty} \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers with the discrete topology. This follows from

$\mathscr{H}_+(A) \cong A \setminus \{0,1\} \cong \mathbb{Z} : f \mapsto f(1/2)$ and $\mathscr{H}_+(X) \cong \prod_{n=1}^{\infty} \mathscr{H}_+(X_n) : f \mapsto (f|_{X_n})_n$.

(ii) If we replace A by $B = \{0\} \cup \{\frac{1}{n} : n \geq 1\}$ and construct a compactum $Y \subset I$ from B as in (i), then $Y \cong X$ but $\mathscr{H}_+(Y) = \{id_Y\}$.

(2) If C is a Cantor set in I, then $\mathscr{H}_+(C)$ has no isolated point.

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