Memoirs of the Faculty of Engineering and Design
Kyoto Institute of Technology 52 (March, 2004)

Period Reproducing Forms and Extremal Length

Dedicated to Professor Masakazu SHIBA on his sixtieth birthday

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(Received September 25, 2003; Accepted December 2, 2003)

Abstract

We shall investigate period reproducing forms on an analytic submanifold in $\mathbf{R}^3$ and give a relation between a period reproducing form and a period reproducing differential of the boundary surface as a Riemann surface. An Accola type theorem for period reproducing forms is valid as in the case of Riemann surface.

Key Words: *Solid torus; harmonic forms; period reproducing differentials; extremal length.*

1. Introduction

Electric and magnetic fields on a solid torus lead us to an image of harmonic forms. Consider a typical solid torus

$$D=\left\{(x,y,z):(x-b\cos\theta)^2+(y-b\sin\theta)^2+z^2\le a^2, 0\le\theta\le 2\pi\right\}$$

and a 2-form $\omega=2\pi(-ydydz+xdzdx)/(x^2+y^2)$ on D. The ω denotes a magnetic field by the electric current uniformly coiled around the surface of D and each magnetic force line is a closed curve. The period reproducing form of the closed curve is $2\pi(-ydx+xdy)/(b-\sqrt{b^2-a^2})(x^2+y^2)$. The extremal length of a family of these magnetic force lines is expected to be $b-\sqrt{b^2-a^2}$. If we deform D a little, then the magnetic field will change and the magnetic force line may not be a closed curve. We intend to define a extremal length of a family of the magnetic force lines on an analytic submanifold in $\mathbf{R}^3$. The extremal length is defined with a small difference from the usual form.

2. Period reproducing forms

Let V be a domain such as a solid torus in the three dimensional Euclidean space $\mathbf{R}^3$, whose boundary consists of analytic surfaces. For a 1-cycle γ, $\sigma(\gamma)=adx+bdy+cdz$ denotes harmonic period reproducing 1-form for γ, which satisfies

$$\int_\gamma \omega=(\omega,\sigma(\gamma))=\iiint_V \omega\wedge{}^*\sigma(\gamma)$$

for every closed one form ω with a finite Dirichlet norm, where

$$*\sigma(\gamma) = adydz + bdzdx + cdxdy.$$

We refer to the connection between $\sigma(\gamma)$ and period reproducing differential of the Riemann surface by conformal structure $\{du + idv\}$ induced from euclidean metric on the surface. Let $\sigma'(\gamma')$ be the period reproducing harmonic differential on the surface for a 1-cycle γ' which is homologous to γ. For any closed differential ω' on the Riemann surface, it holds

$$\int_{\gamma'} \omega' = (\omega', \sigma'(\gamma')) = \iint_{\partial V} \omega' \wedge *'\sigma'(\gamma'),$$

where $*'$ denotes $*$ operator on the Riemann surface, i.e. for $\sigma'(\gamma') = pdu + qdv, *'\sigma'(\gamma') = -qdu + pdv$, $*'\sigma'(\gamma')$ is a differential of the function on the surface deleted γ'. Let the surface be represented as $\{(x,y,z): x = x(u,v), y = y(u,v), z = z(u,v)\}$ and $u = u(x,y,z), v = v(x,y,z)$. Then u and v are real analytic functions which satisfy

$$\begin{aligned} *'\sigma'(\gamma') &= -q\left(u(x,y,z), v(x,y,z)\right)\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) \\ &\quad + p\left(u(x,y,z), v(x,y,z)\right)\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right) \\ &= P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz. \end{aligned}$$

These functions $P(x,y,z), Q(x,y,z)$, and $R(x,y,z)$ are real analytic and are defined on a neighborhood of the surface. Using them, set

$$\sigma'' = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Take C^∞ function f which is 1 on a neighborhood of the surface, and 0 in the part on which they are not defined. Then $f\sigma''$ is defined on $\mathbf{R}^3$. For any closed 1-form ω on $\mathbf{R}^3$,

$$\begin{aligned} (\omega, *d(f\sigma'')) &= \iiint_V \omega \wedge d(f\sigma'') \\ &= -\iiint_V d(\omega \wedge f\sigma'') + \iiint_V d\omega \wedge f\sigma'' \\ &= -\iint_{\partial V} \omega \wedge f\sigma'' = -\iint_{\partial V} \omega \wedge *'\sigma'(\gamma') \\ &= -\int_{\gamma'} \omega = -\int_{\gamma} \omega. \end{aligned}$$

Hence orthogonal projection of $-*d(f\sigma'')$ to harmonic 1-form is $\sigma(\gamma)$.

Note that for any C^∞ function g in a neighborhood of D

$$0 = (dg, \sigma(\gamma)) = \iiint_V dg \wedge *\sigma(\gamma) = \iint_{\partial V} g(*\sigma(\gamma)).$$

Hence $*\sigma(\gamma) = 0$ along ∂V.

3. Extremal length of a curve family

Let V be a solid torus. For a closed curve γ in V, which is not homologous to 0, take a curve family

$$\Gamma = \{\gamma' : \gamma' \text{ is a 1-cycle in } V \text{ which is homologous to } \gamma\}.$$

Extremal length of Γ is defined by

$$\lambda(\Gamma) = \frac{1}{A(\Gamma)}, \text{ where } A(\Gamma) = inf\left\{\iiint_V \rho^2\, dxdydz : \rho \in T\right\},$$

$$T = \{\rho : \rho \text{ is a bounded non-negative Borel measurable density function on } V,$$
$$\text{and satisfies } \int_{\gamma'} \rho\, ds \geq n \text{ for every natural number } n$$
$$\text{and rectifiable } \gamma' \text{ which is homologous to } n\gamma\}.$$

Here we will show the following Accola's theorem.

Theorem 3.1.

$$\lambda(\Gamma) = \|\sigma(\gamma)\|^2.$$

Proof. Now $\sigma(\gamma) = adx + bdy + cdz$ satisfies the following;

$$\|\sigma(\gamma)\|^2 = \iiint_V \sigma(\gamma) \wedge {}^*\sigma(\gamma) = \iiint_V (a^2 + b^2 + c^2)\, dxdydz.$$

And for any rectifiable curve γ' in V

$$\int_{\gamma'} |\sigma(\gamma)| = \int_{\gamma'} \left| a\frac{\partial x}{\partial t} + b\frac{\partial y}{\partial t} + c\frac{\partial z}{\partial t} \right| dt$$
$$\leq \int_{\gamma'} \sqrt{a^2+b^2+c^2} \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}\, dt$$
$$= \int_{\gamma'} \sqrt{a^2+b^2+c^2}\, ds,$$

where ds is the length element of γ'. For $\rho = \sqrt{a^2+b^2+c^2} \big/ \|\sigma(\gamma)\|^2$, it follows that

$$\int_{\gamma'} \rho\, ds \geq \frac{1}{\|\sigma(\gamma)\|^2} \int_{\gamma'} |\sigma(\gamma)| \geq \frac{1}{\|\sigma(\gamma)\|^2} \int_{\gamma'} \sigma(\gamma) = \frac{n}{\|\sigma(\gamma)\|^2} (\sigma(\gamma), \sigma(\gamma)) = n.$$

On the other hand,

$$\iiint_V \rho^2\, dxdydz = \frac{1}{\|\sigma(\gamma)\|^4} \iiint_V \left(a^2+b^2+c^2\right) dxdydz = \frac{1}{\|\sigma(\gamma)\|^2}.$$

Therefore

$$\lambda(\Gamma) = \frac{1}{A(\Gamma)} \geq \|\sigma(\gamma)\|^2.$$

For $p \in V$, put

$$\alpha_\theta = \left\{ q : \frac{1}{\|\sigma(\gamma)\|^2} \int_p^q \sigma(\gamma) = \theta \ mod\ Z \right\}, \text{ where } 0 \le \theta < 1.$$

Any component V_k of $V - \alpha_0$ does not contain a cycle γ. Because $\int_\gamma \sigma(\gamma) = \|\sigma(\gamma)\|^2$ and $V_k \cap \alpha_0 \neq \emptyset$ if it contains γ, this is a contradiction. The $\sigma(\gamma)$ is represented as $\sigma(\gamma) = du_k$ by a function u_k on V_k, such that $u_k = 0$ on α_k, $e = \|\sigma(\gamma)\|^2$ on α'_k, in the case V_k is surrounded by α_k, α'_k, and ∂V. We have

$$\iiint_{V_k} du_k \wedge {*du_k} = \iiint_{V_k} d(u_k(*du_k)) - \iiint_{V_k} u_k d(*du_k)$$

$$= \iint_{\partial V_k} u_k(*du_k) = e \iint_{\alpha'_k} *du_k + \iint_{\partial V \cap \partial V_k} u_k(*du_k) = e \iint_{\alpha'_k} *du_k.$$

Note that

$$0 = (d1, du_k)_{V_k} = \iint_{\partial V_k} *du_k = \iint_{\alpha'_k} *du_k - \iint_{\alpha_k} *du_k + \iint_{\partial V \cap \partial V_k} *du_k.$$

Since $\iint_{\partial V \cap \partial V_k} *du_k = 0$, we h

$$\iint_{\alpha'_k} *du_k = \iint_{\alpha_k} *du_k,$$

$$e = \|\sigma(\gamma)\|^2 = e \sum \iint_{\alpha_k} *du_k = e \iint_{\alpha_0} *\sigma(\gamma).$$

It follows that

$$\iint_{\alpha_\theta} *\sigma(\gamma) = \iint_{\alpha_0} *\sigma(\gamma) = 1.$$

A component of α_0 does not surround a domain nor divide V. Hence α_0 consists of a unique component in solid torus V. We rewrite u_k as u. Along the

$$\alpha_\theta = \{q : u(q) = \theta\} = \{(x(t,s), y(t,s), z(t,s)) : (t,s)\},$$

we have

$$0 = du = u_x dx + u_y dy + u_z dz$$

$$= (u_x x_t + u_y y_t + u_z z_t) dt + (u_x x_s + u_y y_s + u_z z_s) ds,$$

and

$$ax_t + by_t + cz_t = 0, \quad ax_s + by_s + cz_s = 0.$$

The normal vector of the surface is orthogonal to the vector (a, b, c) and is represented as a linear combination of (x_t, y_t, z_t) and (x_s, y_s, z_s). The α_θ is orthogonal to ∂V. Integral curve

$\beta(p_0)$ of

$$\frac{dx}{d\tau} = a\left(x(\tau), y(\tau), z(\tau)\right)$$

$$\frac{dy}{d\tau} = b\left(x(\tau), y(\tau), z(\tau)\right)$$

$$\frac{dz}{d\tau} = c\left(x(\tau), y(\tau), z(\tau)\right)$$

is uniquely determined if it goes through a point $p_i \in \alpha_0$. $\beta(p_0)$ is orthogonal to α_θ. If the point p_0 lies on ∂V, $\beta(p_0)$ lies on ∂V. If p_0 lies inside of V, $\beta(p_0)$ also lies inside of V. Let $p_n \in \alpha_0$ denote the ending point when $\beta(p_0)$ meets α_0 n-times after starting at p_0. Denote this segment by $\beta'_n(p_0)$ and take a curve $\beta''_n(p_0)$ which connects p_n and p_0 on α_0. The curve $\beta'_n(p_0) \cup \beta''_n(p_0) = \beta_n(p_0)$ becomes a closed curve which is homologous to $n\gamma$. For $\rho \in T$

$$n \leq \int_{\beta_n(p_0)} \rho\, ds$$

$$= \int_{\beta'_n(p_0)} \rho\left(x(\tau), y(\tau), z(\tau)\right) \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}\, d\tau + \int_{\beta''_n(p_0)} \rho\, ds,$$

and put

$$M = sup\left\{inf\left\{\int_{\gamma(p,q)} \rho\, ds : \gamma(p,q) \text{ is a curve in } \alpha_0 \text{ which starts from } p \text{ to } q\right\} : p, q \in \alpha_0\right\}.$$

We can choose a curve $\beta''_n(p_0)$ so that

$$\int_{\beta''_n(p_0)} \rho\, ds \leq 2M.$$

On $\beta'_n(p_0)$, put $\theta(\tau) = u(x(\tau), y(\tau), z(\tau))$. From

$$d\theta = \left(u_x x_\tau + u_y y_\tau + u_z z_\tau\right) d\tau = \left(a^2 + b^2 + c^2\right) d\tau,$$

we have

$$n - 2M \leq \int_0^{ne} \frac{\rho}{\sqrt{a^2 + b^2 + c^2}}\, d\theta \leq \sqrt{\int_0^{ne} \frac{\rho^2}{a^2 + b^2 + c^2}\, d\theta}\, \sqrt{\int_0^{ne} d\theta},$$

and

$$(n - 2M)^2 = \int_{\alpha_0} (n - 2M)^2 * du \leq ne \int_{\alpha_0} * du \int_0^{ne} \frac{\rho^2}{a^2 + b^2 + c^2}\, du$$

$$\leq n^2 e \iiint_V \frac{\rho^2}{a^2 + b^2 + c^2}\, du \wedge * du = n^2 e \iiint_V \rho^2\, dxdydz.$$

Since

$$\iiint_V \rho^2\, dxdydz \geq \left(1 - \frac{2M}{n}\right)^2 \frac{1}{e},$$

we have

$$\iiint_V \rho^2\, dxdydz \geq \frac{1}{e} = \frac{1}{\|\sigma(\gamma)\|^2}\,.$$

Therefore

$$\lambda(\Gamma) \leq \|\sigma(\gamma)\|^2.$$

This concludes

$$\lambda(\Gamma) = \|\sigma(\gamma)\|^2.$$

4. Harmonic forms restricted by boundary behavior

Let F^k be the real Hilbert space of square integrable k-forms on V by Dirichlet's inner product. Consider the following subspaces:
F^k_{eo} is a completion of the class of exterior derivatives of $C^\infty(k-1)$-forms with compact support,
$F^k_h = \{\omega \in F^k; \omega$ is a k-form which is orthogonal to $F^k_{eo} + {*F^{3-k}_{eo}}\}$,
$F^k_{he} = \{d\omega \in F^k_h; \omega \in F^{k-1}\}$,
$F^k_{ho} = \{\omega \in F^k_h; \omega$ is orthogonal to ${*F^{3-k}_{he}}\}$.
We call the element of F^k_h harmonic k-form. We have the well-known orthogonal decomposition;

$$F^k = F^k_h + F^k_{eo} + {*F^{3-k}_{eo}}, \quad F^k_h = F^k_{he} + {*F^{3-k}_{ho}} = F^k_{ho} + {*F^{3-k}_{he}}.$$

Let S be an oriented analytic surface in V. We assume that there is a neighborhood in V which is diffeomorphic to $S\times[-1,1]$. The orientation of S is positive for the direction corresponding to vector (0,1).

(I) Suppose that S is compact in V. Let f_s be a C^∞ function on $V-S$ such that $f_s=1$ on the part SV^+ diffeomorphic to $S\times(0,1)$, $f_s=0$ on the part SV^- diffeomorphic to $S\times(-1,0)$, and $f_s=0$ on a neighborhood of the boundary of V. The one form df_s has the following orthogonal decomposition:

$$df_s = \omega_s + \omega_1 + \omega_0, \quad \omega_s \in F^1_{ho}, \quad \omega_1 \in {*F^2_{he}}, \quad \omega_0 \in F^1_{eo}.$$

We have

$$\|\omega_1\|^2 = (df_s, \omega_1) = \iiint_V df_s \wedge {*\omega_1} = \iint_{\partial V} df_s \wedge \sigma_1 = 0,$$

where $d\sigma_1 = {*\omega_1}$. Hence $\omega_1 = 0$. If S divides V, there is a function w_s such that $dw_s = \omega_s \in F^1_{he}$. For $\omega_0 = dw_0$ and any one form $\omega \in {*F^2_{he}} + {*F^2_{eo}}$,

$$0 = (\omega_0, \omega) = \iint_{\partial V} w_0 {*\omega}.$$

It follows that $w_0 = 0$ on the boundary ∂V, $w_s = 1$ on the boundary α of the component which contains SV^-, and $w_s=0$ on the boundary of the component which contains SV^+. For any $\sigma \in F_h^2$,

$$(\sigma, *\omega_s) = \iiint_V dw_s \wedge \sigma = \iiint_V d(w_s\sigma) = \iint_\alpha \sigma = \iint_S \sigma.$$

Therefore $*dw_s \in F_h^2$ is the harmonic period reproducing 2-form of S.

(II) When S is not compact in V and doesn't divide V, let g_s be a C^∞ function on $V-S$ such that $g_s=1$ on SV^+, $g_s=0$ on SV^- and $*dg_s=0$ along ∂V. The one form dg_s has the following orthogonal decomposition:

$$dg_s = \sigma_s + \sigma_1 + \sigma_0,\ \sigma_s \in *F_{ho}^2,\ \sigma_1 \in F_{he}^1,\ \sigma_0 \in F_{eo}^1.$$

There are functions u_1 and u_0 on $V-S$ such that $du_1=\sigma_1, du_0=\sigma_0$. Set $u_s=g_s-u_1-u_0$, then u_s is a function on $V-S$ and $du_s=\sigma_s$. It satisfies that

$$\lim_{SV^+ \ni a' \to a} u_s(a') - \lim_{SV^- \ni a' \to a} u_s(a') = 1.$$

Let p be fixed and consider the following surface

$$\left\{ q;\ \int_p^q \sigma_s \in Z \right\} = \cup \beta_i,$$

where the integral path is in $V-S$ and every β_i denotes a component with the natural orientation. Set $V - \cup\beta_i = \cup V_j$, where every V_j denotes a component. The boundary of V_j consists of some $\{\beta_i\}, \{-\beta_{i'}\}$ and $\partial V \cup \partial V_j$. For any $dw \in F_{he}^1 + F_{eo}^1$,

$$0 = (dw, \sigma_s) = \iint_{\partial V} w(*\sigma_s).$$

Hence $*\sigma_s=0$ on ∂V. We have

$$\|\sigma_s\|_{V_j}^2 = \iiint_{V_j} d(u_s(*\sigma_s)) = \iint_{\partial V \cap \partial V_j + \Sigma\beta_i - \Sigma\beta_{i'}} u_s(*\sigma_s) = \iint_{\Sigma\beta_i} *\sigma_s.$$

For any $du \in F_h^1 \cap F_{he}^1(V-S)$, set

$$P(u) = \lim_{SV^+ \ni a' \to a} u(a') - \lim_{SV^- \ni a' \to a} u(a').$$

This $P(u)$ is the period of du along the closed curve γ in $V-S$ which connects both sides of S. We have

$$(du, \sigma_s) = \iiint_{V-S} du \wedge *\sigma_s = \iint_{\partial(V-S)} u(*\sigma_s) = P(u) \iint_S *\sigma_s$$

and further

$$\iint_S *\sigma_s = \|\sigma_s\|_V^2 = \iint_{\Sigma\beta_i} *\sigma_s,$$

hence $\sigma_s/\|\sigma_s\|^2$ is the period reproducing form of $F_h^1 \cap F_{he}^1(V-S)$.

For $x=h$ or $x=ho$, let

$\Gamma(S;x)=\{S';S'$ is a cycle consisting of a finite number of oriented surfaces in V,

$$\text{and satisfies } \iint_{S'} \omega = \iint_{S} \omega \text{ for any } \omega \in F_x^2\}.$$

Non negative Borel measurable density function ρ in V is admissible for $\Gamma(S;x)$ if it satisfies that for any $S' \in \Gamma(S;x)$

$$\iint_{S'} \rho\, dS' \geq 1,$$

where dS' is the area element of the surface S'. Extremal length of $\Gamma(S;x)$ is defined by

$$\lambda(\Gamma(S;x)) = 1/inf\{\iiint_V \rho^2\, dV;\rho \text{ is admissible for } \Gamma(S;x)\},$$

where dV is the volume element.

We put $\tau_s=\omega_s$ if $x=h$ in case (I) and $\tau_s=\sigma_s$ if $x=ho$ in case (II). Take the following density function

$$\rho_{s,x}=\frac{1}{\|\tau_s\|^2}\sqrt{\frac{\tau_s \wedge * \tau_s}{dV}}.$$

We have the following.

Theorem 4.1.

$$\lambda(\Gamma(S;x)) = \|\tau_s\|^2.$$

Proof. For $S' \in \Gamma(S;x)$,

$$\iint_{S'} \rho_{s,x}\, dS' = \frac{1}{\|\tau_s\|^2}\iint_{S'}\sqrt{\frac{\tau_s \wedge * \tau_s}{dV}}\, dS'$$

$$\geq \frac{1}{\|\tau_s\|^2}\iint_{S'} * \tau_s = \frac{1}{\|\tau_s\|^2}\iint_{S} * \tau_s = 1.$$

Thus $\rho_{s,x}$ is admissible for $S' \in \Gamma(S;x)$, and

$$\lambda(\Gamma(S;x)) \geq \frac{1}{\iiint_V \rho_{s,x}^2\, dV} = \|\tau_s\|^2.$$

Set $\alpha_{t,x}=\{p;v_s(p)=t\}$, where $x=h, v_s=w_s$ in case (I) and $x=ho, v_s=u_s$ in case (II). Any $\alpha_{t,x} \in \Gamma(S;x)$ satisfies

$$\iint_{\alpha_{t,x}} \rho\, d\alpha_{t,x} \geq 1, \text{ for any admissible } \rho \text{ for } \Gamma(S;x).$$

We have

$$dv_s = \frac{\partial v_s}{\partial n} dn, \text{ on } \alpha_{t,x},$$

where dn is the normal unit vector on the surface $\alpha_{t,x}$. Using $dV = d\alpha_{t,x} \wedge dn$, note that

$$*\tau_s = \frac{\partial v_s}{\partial n} d\alpha_t, \quad *\tau_s \wedge \tau_s = (\frac{\partial v_s}{\partial n})^2 d\alpha_{t,x} \wedge dn, \quad \rho_{s,x} = \frac{1}{\|\tau_s\|^2} \left|\frac{\partial v_s}{\partial n}\right|.$$

Hence

$$\begin{aligned}\iiint_V \rho\rho_{s,x} dV &= \frac{1}{\|\tau_s\|^2} \iiint_{V-S} \rho \left|\frac{\partial v_s}{\partial n}\right| d\alpha_{t,x} \wedge dn \\ &= \frac{1}{\|\tau_s\|^2} \iiint_{V-S} \rho\, d\alpha_{t,x} \wedge dv_s \\ &= \frac{1}{\|\tau_s\|^2} \int_0^1 dt \iint_{\alpha_{t,x}} \rho\, d\alpha_{t,x} \geq \frac{1}{\|\tau_s\|^2}.\end{aligned}$$

On the other hand, by Schwarz's inequality

$$\begin{aligned}\iiint_V \rho\rho_{s,x} dV &\leq \sqrt{\iiint_V \rho^2 dV \iiint_V \rho_{s,x}^2 dV} \\ &= \frac{1}{\|\tau_s\|} \sqrt{\iiint_V \rho^2 dV}.\end{aligned}$$

Therefore

$$\iiint_V \rho^2 dV \geq \frac{1}{\|\tau_s\|^2}$$

and

$$\lambda(\Gamma(S;x)) = \|\tau_s\|^2.$$

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