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A Sequence of Behavior Spaces and the Structure of Its Convergent Space

Dedicated to Emeritus Professor Yukio KUSUNOKI in Kyoto University on his eightieth birthday

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Abstract

The concept of behavior spaces introduced by Shiba plays an important role of systematic investigation of abelian differentials on an open Riemann surface. A behavior space consists of holomorphic differentials which satisfy a certain period condition and boundary behavior. For a Riemann surface of infinite genus, the existence of behavior spaces with a general period condition is not guaranteed. For the sake of this thesis we consider a sequence of behavior spaces and the convergence. The result is a steppingstone to the thesis.

Key Words: *Riemann surface; harmonic differentials; period conditions; behavior spaces.*

1. Introduction

Let $\Lambda = \Lambda(R)$ be a real Hilbert space which consists of square integrable complex differentials on a Riemann surface R. Its inner product is given as follows:

$$< \omega, \sigma >= Real\ part\ of \iint_R \omega \wedge {}^*\bar{\sigma} = \Re(\omega, \sigma),$$

where $\bar{\sigma}$ is the complex conjugate differential of σ, ${}^*\bar{\sigma}$ is the conjugate differential of $\bar{\sigma}$, (ω, σ) denotes the integral itself of above second expression which means the complex inner product and $\Re(\omega, \sigma)$ means its real part. We use the following subspaces.

$\Lambda_h = \{\lambda \in \Lambda : \lambda \text{ is harmonic}\}$,
$\Lambda_{eo} = \{\lambda \in \Lambda : \lambda \text{ is orthogonal to } \Lambda_h \text{ and a closed differential}\}$,
$\Lambda_{hse} = \{\lambda \in \Lambda_h : \int_\gamma \lambda = 0 \text{ for any dividing cycle } \gamma\}$,
$\Lambda_{he} = \{\lambda \in \Lambda_h : \lambda \text{ is exact}\}$, $\Lambda_{ho} = {}^*\Lambda_{he}^\perp$, $\Lambda_{hm} = {}^*\Lambda_{hse}^\perp$,

where $\Lambda_\chi^\perp$ is the orthogonal complement of Λ_χ in Λ_h and ${}^*\Lambda_\chi = \{ {}^*\omega : \omega \in \Lambda_\chi\}$.

For $z \in \mathbf{C}$ we put $z\Lambda_\chi = \{z\omega : \omega \in \Lambda_\chi\}$. Then $(z\Lambda_\chi)^\perp = z\Lambda_\chi^\perp$.
Note the following orthogonal decompositions:

$$\Lambda_h = \Lambda_{he} \oplus {}^*\Lambda_{ho} = \Lambda_{hse} \oplus {}^*\Lambda_{hm} = {}^*\Lambda_{he} \oplus \Lambda_{ho} = {}^*\Lambda_{hse} \oplus \Lambda_{hm}.$$

Let

$$\Gamma_h = \{\lambda \in \Lambda_h : \lambda \text{ is real}\},\ \Gamma_{hse} = \Gamma_h \cap \Lambda_{hse},$$

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$$\Gamma_{he} = \Gamma_h \cap \Lambda_{he},\ \Gamma_{ho} = {}^*\Gamma_{he}^{\perp},\ \Gamma_{hm} = {}^*\Gamma_{hse}^{\perp},$$

where $\Gamma_\chi^\perp$ is the orthogonal complement of Γ_χ in Γ_h and ${}^*\Gamma_\chi = \{ {}^*\omega; \omega \in \Gamma_\chi\}$.

Let $\{R_m\}$ be a canonical exhaustion of R, and $\{A_j, B_j\}$ be canonical homology basis of Ahlfors type associated to $\{R_m\}$, i.e., B_j crosses A_j from left to right, and furthermore, whose restriction to $(R_{m+1} - R_m)$ is the canonical homology basis of $(R_{m+1} - R_m)$ with $mod\ \partial(R_{m+1} - R_m)$. Let $g(\leq \infty)$ be the genus of R. We classify the set of numbers $\boldsymbol{J} = \{1, 2, ..., g\}$ ($\boldsymbol{J}$ is the all natural numbers **N** if $g = \infty$) to k classes $\{\boldsymbol{J}_i\}(k \leq \infty)$, i.e., $\boldsymbol{J} = \cup_{i=1}^k \boldsymbol{J}_i$, $\boldsymbol{J}_i \cap \boldsymbol{J}_j = \emptyset$ for $i \neq j$. Let $L = \{L_j\}$ be a family of lines in the complex plane, on which zero lies. Assume that $L_j = L_k$ if $j, k \in \boldsymbol{J}_i$. A subspace $\Lambda_\chi = \Lambda_\chi(\boldsymbol{J}, L)$ of Λ_h is called a behavior space associated to $\boldsymbol{J}$, L if

$$1)\ \Lambda_\chi = i\, {}^*\Lambda_\chi^\perp \subset Closure\ of \{\Lambda_{he} + \Lambda_{ho}\},$$

$$2)\ \int_{A_j} \lambda \in L_i,\ \int_{B_j} \lambda \in L_i,\ for\ j \in \boldsymbol{J}_i,\ \lambda \in \Lambda_\chi.$$

Our subject is the existence of such a behavior space for arbitrary given $L = \{L_i\}$.

2. Canonical behavior spaces

At first we introduce a typical behavior space which is called a canonical behavior space. We consider the following subspaces of Λ_h :

$$S_m(L) = \{\sum (a_j e^{i\theta_j} \sigma_{A_j} + b_j e^{i\theta_j} \sigma_{B_j})\ ; a_j, b_j \in \mathbf{R},\ e^{i\theta_j} \in L_j,\ A_j,\ B_j \subset R_m\},$$

$$\underline{\Lambda}_{cm} = \underline{\Lambda}_{cm}(L) = Closure\ of \{\Gamma_{he} \oplus i(\Gamma_{he} \cap \Gamma_{ho}) + S_m(L)\}$$

$$S^m = \{\sum (a_j \sigma_{A_j} + b_j \sigma_{B_j})\ ; a_j, b_j \in \mathbf{R}, A_j,\ B_j \subset (R - R_m)\},$$

$$\Lambda_{cm} = \Lambda_{cm}(L) = Closure\ of \{\Gamma_{he} \oplus i(\Gamma_{he} \cap \Gamma_{ho}) + S_m(L) + iS^m\},$$

where **R** denotes the real numbers, $\sigma_\gamma \in \Gamma_{ho}$ is the period reproducing differential of a closed curve $\gamma \subset R$, i.e.,

$$\int_\gamma \lambda = (\lambda,\ {}^*\sigma_\gamma).$$

Let $\boldsymbol{J}_i^m = \boldsymbol{J}_i \cap \{j : A_j \subset R_m\}, \boldsymbol{J}_{m\infty} = \{j; A_j \subset (R - R_m)\}$, and $L_{m\infty}$ be the imaginary axis. Set $\boldsymbol{J}^m = \cup_{i=1}^k \boldsymbol{J}_i^m \cup \boldsymbol{J}_{m\infty}$, $L^m = \{L_i, L_{m\infty}; \boldsymbol{J}_i^m \neq \emptyset\}$.

Proposition 2.1. Λ_{cm} is a behavior space associated to $\boldsymbol{J}^m$, L^m.

Proof. Since Γ_{he} and $S_m(L)$ are orthogonal to ${}^*(\Gamma_{ho} + S_m(L) + S^m)$ and ${}^*S^m$ respectively, Λ_{cm} is orthogonal to $i\, {}^*\Lambda_{cm}$. We set

$$\tilde{\Lambda}_{cm} = \{\lambda \in Closure\ of \{\Gamma_{he} + \Gamma_{ho}\} + i\Gamma_{ho}\ ; \int_{A_j} \lambda, \int_{B_j} \lambda \in L_i,\ for\ j \in \boldsymbol{J}_i,\ A_j,\ B_j \subset R_m$$

$$\int_{A_j} \lambda, \int_{B_j} \lambda \in L_{m\infty},\ for\ j \in \boldsymbol{J}_{m\infty},\ A_j,\ B_j \subset R - R_m\},$$

and note that

$$\Lambda_{cm} + i\,{}^*\Lambda_{cm} = Closure\ of\{\Gamma_{he} + {}^*(\Gamma_{he} \cap \Gamma_{ho}) + {}^*S^m + S_m(L)$$
$$+i({}^*\Gamma_{he} + (\Gamma_{he} \cap \Gamma_{ho}) + S^m) + i\,{}^*S_m(L)\}.$$

The following is well known.

Lemma 2.1. For C^1-differentials $\omega \in \Lambda_{hse} + \Lambda_{eo}, \sigma \in \Lambda_h + \Lambda_{eo}$

$$< \omega, {}^*\sigma >= \lim_{k\to\infty} \Re[\sum_{A_j,B_j\in R_k} (\int_{A_j} \omega \int_{B_j} \bar{\sigma} - \int_{B_j} \omega \int_{A_j} \bar{\sigma}) - \int_{\partial R_k} w\bar{\sigma}],$$

where w is a function on $R_k - \cup(A_j \cup B_j)$ satisfying $\omega = dw$.

For $\omega, \sigma \in \tilde{\Lambda}_{cm}$

$$< \omega, i\,{}^*\sigma >= \lim_{k\to\infty} \Re[-\sum_{A_j,B_j\in R_k} i(\int_{A_j} \omega \int_{B_j} \bar{\sigma} - \int_{B_j} \omega \int_{A_j} \bar{\sigma}) + i\int_{\partial R_k} w\bar{\sigma}]$$

$$= \lim_{k\to\infty} -\int_{\partial R_k} \Re w \Im\bar{\sigma} - \int_{\partial R_k} \Im w \Re\bar{\sigma},$$

where $\Im *$ denotes the imaginary part of $*$. Since $\Re\omega$ and $\Re\bar{\sigma}$ are exact on $R - R_m$, there are C^∞-functions $\tilde{w}$, and $\tilde{s}$ on R, which satisfy

$$\tilde{w} = \tilde{s} = 0 \text{ on } R_m,\ d\tilde{w} = \Re\omega,\ d\tilde{s} = \Re\bar{\sigma} \text{ on } R - R_{m+1}.$$

Hence

$$\lim_{k\to\infty} -\int_{\partial R_k} \Re w \Im\bar{\sigma} = \lim_{k\to\infty} -\int_{\partial R_k} \tilde{w}\Im\bar{\sigma} =< d\tilde{w}, i\,{}^*\sigma >= 0,$$

$$\lim_{k\to\infty} -\int_{\partial R_k} \Im w \Re\bar{\sigma} =< -i\omega, {}^*d\tilde{s} >= 0,$$

and

$$< \omega, i\,{}^*\sigma >= 0,\ \tilde{\Lambda}_{cm} \perp i\,{}^*\tilde{\Lambda}_{cm}.$$

It is clear that $\Lambda_{cm} \subset \tilde{\Lambda}_{cm}$, so if ω is orthogonal to $i\,{}^*\Lambda_{cm}$, then

$$\Re(-ie^{-i\theta_j} \int_{A_j} \omega) =< \omega, ie^{i\theta_j}\,{}^*\sigma_{Aj} >= 0.$$

It follows that $\int_{A_j} \omega \in L_j$ and similarly $\int_{B_j} \omega \in L_j$. The ω is also orthogonal to ${}^*(\Gamma_{he} \cap \Gamma_{ho})$ and $i\,{}^*\Gamma_{he}$, hence $\omega \in Closure\ of\{\ \Gamma_{he} + \Gamma_{ho}\ \} + i\Gamma_{ho}$. We have $\omega \in \tilde{\Lambda}_{cm}$, i.e., $i\,{}^*\Lambda_{cm}^{\perp} \subset \tilde{\Lambda}_{cm}$, and $i\,{}^*\tilde{\Lambda}_{cm}^{\perp} \subset \Lambda_{cm}$. Therefore we get

$$i\,{}^*\tilde{\Lambda}_{cm}^{\perp} \subset \Lambda_{cm} \subset i\,{}^*\Lambda_{cm}^{\perp} \subset \tilde{\Lambda}_{cm} \subset i\,{}^*\tilde{\Lambda}_{cm}^{\perp},$$

$$\Lambda_{cm} = i\,{}^*\Lambda_{cm}^{\perp} = \tilde{\Lambda}_{cm} = i\,{}^*\tilde{\Lambda}_{cm}^{\perp}.$$

This shows that Λ_{cm} is a behavior space, which we call the canonical behavior space. □

When a Riemann surface R belongs to the class O'' (cf.[K1]), we know

$$\lim_{k\to\infty} \int_{\partial R_k} w\bar{\sigma} = 0, \text{ for } C^1\text{-differentials } \omega = dw \in \Lambda_{hse} + \Lambda_{eo}, \sigma \in \Lambda_h + \Lambda_{eo}.$$

Then we have a general Riemann's period relation

$$<\omega,\ {}^*\sigma>=\lim_{k\to\infty}\sum_{A_j,B_j\in R_k}\Re\Big(\int_{A_j}\omega\int_{B_j}\bar{\sigma}-\int_{B_j}\omega\int_{A_j}\bar{\sigma}\Big).$$

The following space

$$\tilde{\Lambda}_{cJ}=\{\lambda\in Closure\ of\{\Gamma_{he}+\Gamma_{ho}\}+i\Gamma_{ho}\ ;\int_{A_j}\lambda,\int_{B_j}\lambda\in L_i,\ for\ j\in \boldsymbol{J}_i\}$$

becomes a behavior space. Because $\tilde{\Lambda}_{cJ}\subset i\,{}^*\tilde{\Lambda}_{cJ}^{\perp}$ and $i\,{}^*\Lambda_{cJ}^{\perp}\subset\tilde{\Lambda}_{cJ}$ are shown in a similar way mentioned above.

3. Convergence of a sequence of differentials

We will use three types of convergence in Λ_h. For a sequence $\{\lambda_n\in\Lambda\}_{n=1}^{\infty}$, we denote
(i) the norm convergence by

$$\text{s-}\lim_{n\to\infty}\lambda_n=\lambda,\ \text{i.e.,}\ \lim_{n\to\infty}||\lambda_n-\lambda||=0,$$

(ii) the weak convergence by

$$\text{w-}\lim_{n\to\infty}\lambda_n=\lambda,\ \text{i.e.,}\ \lim_{n\to\infty}<\lambda_n,\omega>=<\lambda,\omega>\ \text{for each}\ \omega\in\Lambda_h,$$

and (iii) the uniformly bounded convergence on every compact set by

$$\text{BW-}\lim_{J\ni j\to\infty}\lambda_j=\lambda,\ \text{i.e., (a)}\lim_{J\ni j\to\infty}||\lambda_j-\lambda||_{R_m}=0\ \text{for every}\ R_m,$$

and (b) there exists a constant M such that $||\lambda_j||\le M$ for every $j\in J$, where J is a subset of $\mathbf{N}$.

We know

Lemma 3.1. *When a sequence $\{\lambda_n\}\subset\Lambda_h$ is uniformly bounded, i.e., $||\lambda_n||\le M$ for any n, there exists a subsequence $J=\{n_j\}$ so that $\{\lambda_{n_j}\}$ is uniformly convergent on every compact set, i.e.,*

$$\{\lambda_n\in\Lambda_h\}_{n=1}^{\infty},\ ||\lambda_n||\le M\Rightarrow^{\exists}J\subset\mathbf{N}\ \text{s.t.}\ \text{BW-}\lim_{J\ni j\to\infty}\lambda_j=\omega.$$

Proof. Take a family of local disks $V_i=V(a_i)$ center at a_i, which covers $\overline{R_m}$, i.e., $\bigcup_{i=1}^{n(m)}V_i\supset\overline{R_m}$. Since $||\lambda_n||_{V_i}\le M$, there exist a harmonic function h_{ni} on V_i such that $h_{ni}(a_i)=0,\lambda_n=dh_{ni}$. We can choose a subsequence $J(1)\subset N$ such that $\text{BW-}\lim_{J(1)\ni j\to\infty}dh_{j1}=dh^1$ on V_1. In such a way we can choose $J(i)\subset J(i-1)$ such that $\text{BW-}\lim_{J(i)\ni j\to\infty}dh_{ji}=dh^i$ on V_i. Hence there is a harmonic differential ω_m on R_m such that

$$\text{BW-}\lim_{J(n(m))\ni j\to\infty}dh_{ji}=\omega_m\ \text{on}\ R_m\cap V_i.$$

Put $J=\{j_{mm}\}_{m=1}^{\infty}$, where $J(n(k))=\{j_{k\ell}\}_{\ell=1,2,\dots}$. Then there is a harmonic differential $\omega=\omega_m$ on R_m such that $\text{BW-}\lim_{J\ni j\to\infty}\lambda_j=\omega$ on R, $\omega=\omega_m$ on R_m. □

Lemma 3.2. *Let* $\{\lambda_j\} \subset \Lambda_h$ *satisfy* $\overline{\lim_{j\to\infty}} ||\lambda_j|| = \alpha$. *If* BW-$\lim_{j\to\infty} \lambda_j = \lambda$, *then* $||\lambda|| \le \alpha$.

Proof. Since $||\lambda||_{R_m} = \lim_{j\to\infty} ||\lambda_j||_{R_m} \le \alpha$, we have $||\lambda||_R \le \alpha$. □

Lemma 3.3. *Let* BW-$\lim_{j\to\infty} \lambda_j = \lambda$, $\lim_{j\to\infty} ||\lambda_j|| = ||\lambda||$. *Then* s-$\lim_{j\to\infty} \lambda_j = \lambda$.

Proof. For any $\varepsilon > 0$, there exists m such that

$$||\lambda||_R^2 - ||\lambda||_{R_m}^2 = ||\lambda||_{R-R_m}^2 < \varepsilon.$$

Then

$$\overline{\lim_{j\to\infty}} ||\lambda_j||_{R-R_m}^2 = \overline{\lim_{j\to\infty}} (||\lambda_j||_R^2 - ||\lambda_j||_{R_m}^2) = ||\lambda||_R^2 - ||\lambda||_{R_m}^2 < \varepsilon,$$

$$\overline{\lim_{j\to\infty}} ||\lambda_j - \lambda||^2 = \overline{\lim_{j\to\infty}} (||\lambda_j - \lambda||_{R_m}^2 + ||\lambda_j - \lambda||_{R-R_m}^2)$$

$$\le \overline{\lim_{j\to\infty}} (||\lambda_j||_{R-R_m} + ||\lambda||_{R-R_m})^2 < 4\varepsilon.$$

Therefore $\lim_{j\to\infty} ||\lambda_j - \lambda||^2 = 0$. □

Lemma 3.4. *Let* $\{\lambda_n\}, \{\sigma_n\} \subset \Lambda_h$ *and they satisfy* BW-$\lim_{n\to\infty} \lambda_n = \lambda$, s-$\lim_{n\to\infty} \sigma_n = \sigma$. *Then*

$$\lim_{n\to\infty} < \lambda_n, \sigma_n > = < \lambda, \sigma >.$$

Proof. When $||\lambda_n|| \le K$, we have

$$\lim_{n\to\infty} | < \lambda_n, \sigma_n > - < \lambda, \sigma > | = \lim_{n\to\infty} | < \lambda_n, \sigma_n - \sigma > + < \lambda_n - \lambda, \sigma > |$$

$$= \lim_{n\to\infty} \{||\lambda_n|| \, ||\sigma_n - \sigma|| + | < \lambda_n - \lambda, \sigma >_{R_m} | + | < \lambda_n - \lambda, \sigma >_{R-R_m} |\}$$

$$\le \lim_{n\to\infty} ||\lambda_n - \lambda||_{R-R_m} ||\sigma||_{R-R_m} \le 2K||\sigma||_{R-R_m}.$$

Hence

$$\lim_{n\to\infty} | < \lambda_n, \ \sigma_n > - < \lambda, \ \sigma > | = 0.$$ □

Lemma 3.5. *Let* $\{\lambda_n\} \subset \Lambda_h$ *with* $sup_n ||\lambda_n|| \le M < \infty$. *If* $\lambda_J = \lambda$ *for any* $J \subset \mathbf{N}$ *so that*

$$\text{BW-}\lim_{J\ni j\to\infty} \lambda_j = \lambda_J, \text{ then BW-}\lim_{n\to\infty} \lambda_n = \lambda.$$

Proof. If λ_n do not converge to λ bounded weakly, there exists R_m such that $\overline{\lim}_{n\to\infty} ||\lambda_n - \lambda||_{R_m} = \alpha(m) > 0$. Then there is J such that $\lim_{J\ni j\to\infty} ||\lambda_j - \lambda||_{R_m} = \alpha(m)$. By Lemma 3.1, there exists $K \subset J$ such that BW-$\lim_{K\ni k\to\infty} \lambda_k = \lambda_K$. Then $\lambda_K = \lambda$. On the other hand

$$0 = \overline{\lim_{K\ni k\to\infty}} ||\lambda_k - \lambda||_{R_m} = \alpha(m).$$

This is a contradiction. □

4. General behavior spaces

A subspace Λ_χ of Λ_h is said to be a general behavior space if the orthogonal complement of Λ_χ in Λ_h is $i\,{}^*\Lambda_\chi$. For a sequence of general behavior spaces $\{\Lambda_n\}$ we consider the following subspaces:

$$\Lambda_s = \{\lambda \in \Lambda_h : {}^\exists\lambda_n \in \Lambda_n \ s.t. \ \text{s-}\lim_{n\to\infty} \lambda_n = \lambda\},$$
$$\tilde{\Lambda}_{bw} = \{\lambda \in \Lambda_h : {}^\exists J \subset \mathbf{N},\ \lambda_j \in \Lambda_j \ s.t. \ \text{BW-}\lim_{J\ni j\to\infty} \lambda_j = \lambda\},$$
$$\tilde{\Lambda}'_{bw} = \{\sum_{i=1}^{\ell} c_i\,\lambda_i : \lambda_i \in \tilde{\Lambda}_{bw}, c_i \in \mathbf{R}\},$$
$$\Lambda_{bw} = Closure\ of\ (\tilde{\Lambda}'_{bw}).$$

Proposition 4.1.

$$\Lambda_{bw} \oplus i\,{}^*\Lambda_s = \Lambda_h.$$

Proof. For $\lambda \in \tilde{\Lambda}_{bw}$, $\sigma \in \Lambda_s$, by definition, there exist J and $\lambda_j \in \Lambda_j$ such that $\text{BW-}\lim_{J\ni j\to\infty} \lambda_j = \lambda$, and $\sigma_n \in \Lambda_n$ such that $\text{s-}\lim_{n\to\infty} \sigma_n = \sigma$. By Lemma 3.4

$$< \lambda, i\,{}^*\sigma >= \lim_{J\ni j\to\infty} < \lambda_j, i\,{}^*\sigma_j >= 0.$$

Hence $\tilde{\Lambda}_{bw}$ is orthogonal to $i\,{}^*\Lambda_s$. Therefore $\tilde{\Lambda}'_{bw}$, Λ_{bw} are also orthogonal to $i\,{}^*\Lambda_s$. We get $\Lambda_{bw}^\perp \supset i\,{}^*\Lambda_s$.

For $\sigma \in \Lambda_h$ with $< \sigma, i\,{}^*\lambda >= 0$ $(\lambda \in \Lambda_{bw})$, we can take differentials $\sigma_n, \tau_n \in \Lambda_n$ such that $\sigma_n + i\,{}^*\tau_n = \sigma$. Then $||\sigma_n||^2 + ||i\,{}^*\tau_n||^2 = ||\sigma||^2$. For any subsequence J which satisfy $\text{BW-}\lim_{J\ni j\to\infty} \sigma_j = \sigma_J$, we get $\text{BW-}\lim_{J\ni j\to\infty} \tau_j = i\,{}^*(\sigma - \sigma_J) \in \Lambda_{bw}$. Hence $\sigma - \sigma_J \in i\,{}^*\Lambda_{bw}$ and $< \sigma,\ \sigma - \sigma_J >= 0$. By Lemma 3.2, $||\sigma_J||^2 \le ||\sigma||^2$, and

$$||\sigma||^2 =< \sigma, \sigma_J >\le ||\sigma||\,||\sigma_J|| \le ||\sigma||^2, \text{ i.e., } ||\sigma_J|| = ||\sigma||.$$

We have

$$||\sigma||^2 = ||\sigma_J||^2 = ||\sigma_J - \sigma + \sigma||^2 = ||\sigma_J - \sigma||^2 + ||\sigma||^2 \le ||\sigma||^2.$$

Hence $||\sigma_J - \sigma|| = 0$, and $\sigma_J = \sigma$. By Lemma 3.5 we get $\text{BW-}\lim_{n\to\infty} \sigma_n = \sigma$. It follows that

$$||\sigma|| \ge \overline{\lim_{n\to\infty}}||\sigma_n|| \ge \underline{\lim_{n\to\infty}}\,||\sigma_n|| \ge \underline{\lim_{n\to\infty}}\,||\sigma_n||_{R_m} = ||\sigma||_{R_m}.$$

This shows that $\lim_{n\to\infty} ||\sigma_n|| = ||\sigma||$, and $\lim_{n\to\infty} \sigma_n = \sigma \in \Lambda_s$. It leads to

$$i\,{}^*\Lambda_{bw}^\perp \subset \Lambda_s, \text{ i.e., } \Lambda_{bw}^\perp \subset i\,{}^*\Lambda_s, \text{ and } \Lambda_{bw}^\perp = i\,{}^*\Lambda_s. \qquad \square$$

Corollary 4.1.

$$\Lambda_h = \Lambda_s \oplus i\,{}^*\Lambda_s \oplus \Lambda_{bw} \cap i\,{}^*\Lambda_{bw}.$$

Proof. $\Lambda_s \subset \Lambda_{bw}$, and $(\Lambda_s \oplus i\,{}^*\Lambda_s)^\perp = \Lambda_s^\perp \cap i\,{}^*\Lambda_s^\perp = \Lambda_{bw} \bigcap i\,{}^*\Lambda_{bw}$. $\square$

Here we explain the following two examples. Take an orthonormal basis $\{\varphi_n\}_{n=1,2..}$ of a general behavior space Λ_χ.

Example 1.

A subspace Λ'_χ expanded by $i\,{}^*\varphi_1$ and $\{\varphi_n\}_{n=2,3..}$ becomes a general behavior space clearly. Let $\Lambda_{2n-1} = \Lambda_\chi$, $\Lambda_{2n} = \Lambda'_\chi$ and consider the sequence $\{\Lambda_n\}$ of general behavior spaces. The Λ_{bw} for this $\{\Lambda_n\}$ is the subspace expanded by $i\,{}^*\varphi_1$ and $\{\varphi_n\}_{n=1,2..}$. Then Λ_s is the subspace expanded by $\{\varphi_n\}_{n=2,3..}$ and $\Lambda_{bw} \bigcap i\,{}^*\Lambda_{bw}^{\perp} = \{a\varphi_1 + bi\,{}^*\varphi_1; a, b \in \mathbf{R}\}$.

Example 2.

Let $\Lambda'_{\chi m}$ be the subspace expanded by $i\,{}^*\varphi_m$ and $\{\varphi_k\}_{k\neq m}$, set $\Lambda_{2n-1,m} = \Lambda_\chi$, $\Lambda_{2n,m} = \Lambda'_{\chi m}$. The sequence $\{\Lambda_n\}$ of general behavior spaces is taken as follows:

$$\Lambda_1 = \Lambda_{1,1},\ \Lambda_2 = \Lambda_{1,2},\ \Lambda_3 = \Lambda_{2,1},\ \Lambda_4 = \Lambda_{3,1},\ \Lambda_5 = \Lambda_{2,2},\ ....$$

For this sequence $\{\Lambda_n\}$, $\Lambda_s = \{0\}$ and $\Lambda_{bw} = \Lambda_h$.

5. Decomposition sequences of a holomorphic differential

For a non zero holomorphic differential $\varphi(\in \Lambda_a)$, there exist two differentials $\lambda_n(= \lambda_n(\varphi))$, μ_n in a general behavior space Λ_n such that $\varphi = \lambda_n + i\,{}^*\mu_n$. Since $\varphi = i\,{}^*\varphi$, we have $\varphi = i\,{}^*\lambda_n + \mu_n$ and $\lambda_n = \mu_n$. Therefore

$$\varphi = \lambda_n + i\,{}^*\lambda_n,\quad ||\lambda_n||^2 = \frac{1}{2}||\varphi||^2.$$

Set

$$\Lambda_n(\varphi) = \{\omega \in \Lambda_n :< \omega, \lambda_n >= 0\}.$$

Lemma 5.1. *For $m \geq n$, there exist $a_{mn} = a_{mn}(\varphi) \in \mathbf{R}$, $\mu_n^m = \mu_n^m(\varphi) \in \Lambda_n(\varphi)$, and $\underline{\mu}_m^n = \underline{\mu}_m^n(\varphi) \in \Lambda_m(\varphi)$ such that*

$$\lambda_m = a_{mn}\lambda_n + (1 - a_{mn})i\,{}^*\lambda_n + \mu_n^m - i\,{}^*\mu_n^m,$$

$$\lambda_n = a_{mn}\lambda_m + (1 - a_{mn})i\,{}^*\lambda_m + \underline{\mu}_m^n - i\,{}^*\underline{\mu}_m^n,$$

where

$$||\mu_n^m||^2 = ||\underline{\mu}_m^n||^2 = \frac{a_{mn}(1 - a_{mn})}{2}||\varphi||^2.$$

Proof. For $m \geq n$ we have the orthogonal decompositions of λ_m, λ_n as follows:

$$\lambda_m = a_{mn}\,\lambda_n + b_{mn}i\,{}^*\lambda_n + \mu_n^m + i\,{}^*\sigma_n^m,$$

$$\lambda_n = \underline{a}_{nm}\,\lambda_m + \underline{b}_{nm}i\,{}^*\lambda_m + \underline{\mu}_m^n + i\,{}^*\underline{\sigma}_m^n,$$

where a_{mn}, b_{mn}, $\underline{a}_{nm}$, $\underline{b}_{nm} \in \mathbf{R}$, μ_n^m, $\sigma_n^m \in \Lambda_n(\varphi)$, $\underline{\mu}_m^n$, $\underline{\sigma}_m^n \in \Lambda_m(\varphi)$.

Since

$$\begin{aligned}\varphi &= \lambda_n + i\,{}^*\lambda_n = \lambda_m + i\,{}^*\lambda_m\\ &= (a_{mn} + b_{mn})(\lambda_n + i\,{}^*\lambda_n) + (\mu_n^m + \sigma_n^m) + i\,{}^*(\mu_n^m + \sigma_n^m)\\ &= (\underline{a}_{nm} + \underline{b}_{nm})(\lambda_m + i\,{}^*\lambda_m) + (\underline{\mu}_m^n + \underline{\sigma}_m^n) + i\,{}^*(\underline{\mu}_m^n + \underline{\sigma}_m^n),\end{aligned}$$

we get

$$a_{mn} + b_{mn} = \underline{a}_{nm} + \underline{b}_{nm} = 1,\quad \mu_n^m + \sigma_n^m = \underline{\mu}_m^n + \underline{\sigma}_m^n = 0.$$

From

$$< \lambda_m, \lambda_n >= a_{mn}||\lambda_n||^2 = \underline{a}_{nm}||\lambda_m||^2,\ ||\lambda_n||^2 = \frac{1}{2}||\varphi||^2 = ||\lambda_m||^2,$$

we have $a_{mn} = \underline{a}_{nm}$. It follows that

$$\lambda_m = a_{mn}\lambda_n + (1 - a_{mn})i\,{}^*\lambda_n + \mu_n^m - i\,{}^*\mu_n^m,$$

$$\lambda_n = a_{mn}\lambda_m + (1 - a_{mn})i\,{}^*\lambda_m + \underline{\mu}_m^n - i\,{}^*\underline{\mu}_m^n,$$

$$||\lambda_m||^2 = \{|a_{mn}|^2 + (1 - a_{mn})^2\}||\lambda_n||^2 + 2||\mu_n^m||^2,$$

$$||\lambda_n||^2 = \{|a_{mn}|^2 + (1 - a_{mn})^2\}||\lambda_m||^2 + 2||\underline{\mu}_m^n||^2,$$

$$|||\mu_n^m||^2 = ||\underline{\mu}_m^n||^2 = \tfrac{1}{4}(1 - 1 + 2a_{mn} - 2a_{mn}^2)||\varphi||^2 = \tfrac{a_{mn}(1-a_{mn})}{2}||\varphi||^2. \qquad \square$$

Now a subsequence J of natural number **N** is said to be admissible for φ if it satisfies $\text{BW-}\lim_{J\ni j\to\infty} \lambda_j = \lambda_J$. Then, for $j \in J$

$$< \lambda_j, \lambda_n >= a_{jn}||\lambda_n||^2 = \frac{1}{2}a_{jn}||\varphi||^2,$$

hence by lemma 3.4

$$< \lambda_J, \lambda_n >= \lim_{J\ni j\to\infty} < \lambda_j, \lambda_n >= \lim_{J\ni j\to\infty} a_{jn}\frac{||\varphi||^2}{2}.$$

Here we set

$$a_{Jn} = a_{Jn}(\varphi) = \lim_{J\ni j\to\infty} a_{jn} = \frac{2}{||\varphi||^2} < \lambda_J, \lambda_n > .$$

By Lemma 3.4

$$||\lambda_J||^2 =< \lambda_J, \lambda_J >= \lim_{J\ni j\to\infty} < \lambda_J, \lambda_j >= \frac{||\varphi||^2}{2} \lim_{J\ni j\to\infty} a_{Jj}.$$

And we set

$$a_J = a_J(\varphi) = \lim_{J\ni j\to\infty} a_{Jj} = \frac{2}{||\varphi||^2}||\lambda_J||^2.$$

Note that

$$\lambda_J = \lambda_{J'} \Rightarrow a_{Jn} = a_{J'n}\,,\ a_J = a_{J'}.$$

Next,

$$\text{BW-}\lim_{J\ni j\to\infty}(\mu_n^j - i\,{}^*\mu_n^j) = \text{BW-}\lim_{J\ni j\to\infty}\{\lambda_j - a_{jn}\lambda_n - (1 - a_{jn})i\,{}^*\lambda_n\}$$
$$= \lambda_J - a_{Jn}\lambda_n - (1 - a_{Jn})i\,{}^*\lambda_n,$$

and we set

$$\text{BW-}\lim_{J\ni j\to\infty}(\mu_n^j - i\,{}^*\mu_n^j) = \hat{\mu}_n^J.$$

We set $\hat{\mu}_J^J = \text{BW-}\lim_{J\ni j\to\infty} \hat{\mu}_j^J$, then

$$\hat{\mu}_J^J = \text{BW-}\lim_{J\ni j\to\infty}(\lambda_J - a_{Jj}\lambda_j - (1 - a_{Jj})i\,{}^*\lambda_j) = (1 - a_J)(\lambda_J - i\,{}^*\lambda_J).$$

Similarly

$$\text{BW-}\lim_{J\ni j\to\infty}(\underline{\mu}_j^n - i\,{}^*\underline{\mu}_j^n) = \text{BW-}\lim_{J\ni j\to\infty}\{\lambda_n - a_{jn}\lambda_j - (1 - a_{jn})i\,{}^*\lambda_j\}$$
$$= \lambda_n - a_{Jn}\lambda_J - (1 - a_{Jn})i\,{}^*\lambda_J,$$

we set

$$\text{BW-}\lim_{J\ni j\to\infty}(\underline{\mu}_j^n - i\,{}^*\underline{\mu}_j^n) = \hat{\underline{\mu}}_J^n \in \Lambda_{\bar{a}},$$

where $\Lambda_{\bar{a}}(\subset \Lambda)$ consists of anti-holomorphic differentials. Now for any $K \subset J$ such that $\text{BW-}\lim_{K \ni k \to \infty} \mu_n^k = \mu_n^K$, we have

$$\hat{\mu}_n^J = \text{BW-}\lim_{K \ni k \to \infty} (\mu_n^k - i\,{}^*\mu_n^k) = \mu_n^K - i\,{}^*\mu_n^K.$$

On the other hand, $\hat{\mu}_n^J$ satisfies

$$\hat{\mu}_n^J = \sigma_{nJ} + i\,{}^*\tau_{nJ},\ \ \sigma_{nJ},\ \tau_{nJ}\ \in \Lambda_n(\varphi).$$

We have $\mu_n^K = \sigma_{nJ} \in \Lambda_n(\varphi)$. By Lemma 3.5, we denote the limit $\text{BW-}\lim_{J \ni j \to \infty} \mu_n^j$ by μ_n^J. It follows that $\mu_n^J = \sigma_{nJ} \in \Lambda_n(\varphi)$, $\hat{\mu}_n^J = \mu_n^J - i\,{}^*\mu_n^J$. Similarly there exists $K \subset J$ such that $\text{BW-}\lim_{K \ni k \to \infty} \underline{\mu}_k^n = \underline{\mu}_K^n$, we have $\hat{\underline{\mu}}_J^n = \underline{\mu}_K^n - i\,{}^*\underline{\mu}_K^n$.

These are put in order as follows:

$$\begin{aligned}
&\lambda_J = a_{Jn}\lambda_n + (1 - a_{Jn})i\,{}^*\lambda_n + \hat{\mu}_n^J,\\
&\lambda_n = a_{Jn}\lambda_J + (1 - a_{Jn})i\,{}^*\lambda_J + \hat{\underline{\mu}}_J^n,\\
&i\,{}^*\lambda_J = a_{Jn}i\,{}^*\lambda_n + (1 - a_{Jn})\lambda_n + i\,{}^*\hat{\mu}_n^J,\\
&\lambda_J + i\,{}^*\lambda_J = \lambda_n + i\,{}^*\lambda_n = \varphi,\\
&\lambda_J - i\,{}^*\lambda_J = (2a_{Jn} - 1)(\lambda_n - i\,{}^*\lambda_n) + 2(\mu_n^J - i\,{}^*\mu_n^J),\\
&\lambda_n - i\,{}^*\lambda_n = (2a_{Jn} - 1)(\lambda_J - i\,{}^*\lambda_J) + 2(\underline{\mu}_K^n - i\,{}^*\underline{\mu}_K^n).
\end{aligned}$$

Lemma 5.2.

$$||\hat{\mu}_n^J||^2 = (a_J + 2a_{Jn} - 1 - 2a_{Jn}{}^2)\frac{||\varphi||^2}{2}.$$

Proof. From the representation of λ_J, we have

$$||\lambda_J||^2 = (|a_{Jn}|^2 + (1 - a_{Jn})^2)\frac{||\varphi||^2}{2} + ||\hat{\mu}_n^J||^2.$$

By definition of a_J,

$$||\lambda_J||^2 = a_J\frac{||\varphi||^2}{2}.$$

Hence

$$||\hat{\mu}_n^J||^2 = (a_J + 2a_{Jn} - 1 - 2a_{Jn}{}^2)\frac{||\varphi||^2}{2}.$$

□

Proposition 5.1.

(1) $\frac{1}{2} \le a_J \le 1$,

(2) $< \lambda_J,\ i\,{}^*\lambda_J >= (1 - a_J)\frac{||\varphi||^2}{2}$,

(3) $||\lambda_J - i\,{}^*\lambda_J||^2 = (2a_J - 1)||\varphi||^2$.

Proof. (1) By Lemma 5.2

$$2a_J{}^2 - 3a_J + 1 = \lim_{J \ni j \to \infty}(2a_{Jj}{}^2 - a_J - 2a_{Jj} + 1) = -\frac{2}{||\varphi||^2}\lim_{J \ni j \to \infty}||\hat{\mu}_j^J||^2 \le 0.$$

Hence

$$(2a_J-1)(a_J-1)\le 0,\ \frac{1}{2}\le a_J\le 1.$$

(2) It follows that

$$\begin{aligned}&< \lambda_J,\ i^*\lambda_J >= \lim_{J\ni j\to\infty} < \lambda_J,\ i^*\lambda_j >\\ &= \lim_{J\ni j\to\infty} < a_{Jj}\lambda_j + (1-a_{Jj})i^*\lambda_j + \hat{\mu}_j^J,\ i^*\lambda_j >\\ &= \lim_{J\ni j\to\infty} < (1-a_{Jj})i^*\lambda_j,\ i^*\lambda_j >= (1-a_J)\frac{\|\varphi\|^2}{2}.\end{aligned}$$

(3) It follows that

$$\begin{aligned}&\|\lambda_J - i^*\lambda_J\|^2 =< \lambda_J - i^*\lambda_J,\ \lambda_J - i^*\lambda_J >\\ &= \|\lambda_J\|^2 - 2 < i^*\lambda_J,\ \lambda_J > + \|i^*\lambda_J\|^2 = (2a_J-1)\|\varphi\|^2.\end{aligned}$$

□

Corollary 5.1. *The following three conditions are equivalent*

$$(1)\ a_J(\varphi)=\frac{1}{2} \Longleftrightarrow (2)\ \lambda_J(\varphi)=i^*\lambda_J(\varphi) \Longleftrightarrow (3)\ \lambda_J(\varphi)=\frac{\varphi}{2}.$$

These equivalent conditions shows that the φ *belongs to* $\Lambda_{bw}\bigcap i^*\Lambda_{bw}$.

Conversely, for $\varphi\in\tilde{\Lambda}_{bw}\bigcap i^*\tilde{\Lambda}_{bw}$, let $\varphi=\lambda_n+i^*\lambda_n,\ \lambda_n\in\Lambda_n$. When J is admissible for φ, we can take $\{\sigma_j\}$ which converges to φ bounded weakly. We may assume that λ_j converge to λ_J bounded weakly. For any $\omega\in\Lambda_h$, we have

$$\begin{aligned}&< \varphi,\omega >= \lim_{J\ni j\to\infty} < \sigma_j,\omega >= \lim_{J\ni j\to\infty} < \lambda_j + i^*\lambda_j,\omega >= \lim_{J\ni j\to\infty} < \lambda_j,\omega + i^*\omega >\\ &=< \lambda_J,\omega + i^*\omega >=< \lambda_J + i^*\lambda_J,\omega > .\end{aligned}$$

We get $\varphi=\lambda_J+i^*\lambda_J$.

Lemma 5.3.

$$\|\hat{\underline{\mu}}_J^n\|^2 = \{(2a_J-3){a_{Jn}}^2-(2a_J-3)a_{Jn}+\frac{1}{2}(a_J-1)\}\|\varphi\|^2.$$

Proof. By the representation of λ_n, we have

$$\begin{aligned}\|\hat{\underline{\mu}}_J^n\|^2 &= \|\lambda_n - a_{Jn}\lambda_J - (1-a_{Jn})i^*\lambda_J)\|^2\\ &= \|\lambda_n\|^2 + {a_{Jn}}^2\|\lambda_J\|^2 + (1-a_{Jn})^2\|i^*\lambda_J\|^2\\ &\quad -2a_{Jn} < \lambda_n,\lambda_J > +2a_{Jn}(1-a_{Jn}) < \lambda_J, i^*\lambda_J > -2(1-a_{Jn}) < \lambda_n, i^*\lambda_J >\\ &= \frac{\|\varphi\|^2}{2}\{1+{a_{Jn}}^2a_J+(1-a_{Jn})^2a_J-2{a_{Jn}}^2+2a_{Jn}(1-a_{Jn})(1-a_J)-2(1-a_{Jn})^2\}\\ &= \frac{\|\varphi\|^2}{2}\{(4{a_{Jn}}^2-4a_{Jn}+1)a_J-6{a_{Jn}}^2+6a_{Jn}-1\}.\end{aligned}$$

□

Next let

$$X(\varphi) = \{a_J(\varphi) : J \text{ is admissible for } \varphi\},$$
$$A(J) = \{ \text{ accumulation points of } \{a_{Jn}(\varphi)\}_n\},$$
$$\tilde{X}(\varphi) = \bigcup\{A(J) : J \text{ is admissible for } \varphi\}.$$

Theorem 5.1. *The following three conditions are equivalent*

$$(1)\ \tilde{X}(\varphi) = \{1\} \Longleftrightarrow (2)\ \varphi \in \Lambda_s + i\,{}^*\Lambda_s \Longleftrightarrow (3)\ \textit{there exists } s\text{-}\lim_{n\to\infty} \lambda_n(\varphi).$$

Proof. For a subsequence J admissible for φ we have

$$\lambda_J - \lambda_n = (a_{Jn} - 1)(\lambda_n - i\,{}^*\lambda_n) + \mu_n^J - i\,{}^*\mu_n^J.$$

(1) $\Rightarrow$ From $\lim_{n\to\infty} a_{Jn} = 1$, we have

$$\lim_{n\to\infty} ||\mu_n^J||^2 = \lim_{n\to\infty}(a_J + 2a_{Jn} - 1 - 2a_{Jn}{}^2)\frac{||\varphi||^2}{4} = 0.$$

Hence

$$\lim_{n\to\infty} ||\lambda_J - \lambda_n|| = 0,\ \ \lambda_J \in \Lambda_s, \text{ and } \varphi = \lambda_J + i\,{}^*\lambda_J \in \Lambda_s + i\,{}^*\Lambda_s.$$

(2) $\Rightarrow$ By the representation $\varphi = \sigma + i\,{}^*\tau$ ($\exists^1\sigma,\ \tau \in \Lambda_s$), we have

$$< \sigma,\ i\,{}^*\tau >= 0,\ \varphi = i\,{}^*\varphi = \tau + i\,{}^*\sigma,\ \therefore \sigma = \tau.$$

Then take a sequence $\sigma_n \in \Lambda_n$ such that $\text{s-}\lim_{n\to\infty} \sigma_n = \sigma$, we have

$$0 = \lambda_n - \sigma + i\,{}^*(\lambda_n - \sigma) = \lambda_n - \sigma_n + i\,{}^*(\lambda_n - \sigma_n) + (\sigma_n - \sigma) + i\,{}^*(\sigma_n - \sigma).$$

Hence

$$(\lambda_n - \sigma_n) + i\,{}^*(\lambda_n - \sigma_n) = -\{(\sigma_n - \sigma) + i\,{}^*(\sigma_n - \sigma)\}.$$

It follows that

$$||\lambda_n - \sigma_n||^2 \le 2||\sigma_n - \sigma||^2,$$

and

$$||\lambda_n - \lambda_m|| \le ||\lambda_n - \sigma_n|| + ||\sigma_m - \lambda_m|| + ||\sigma_n - \sigma_m||$$
$$\le \sqrt{2}(||\sigma_n - \sigma|| + ||\sigma_m - \sigma||) + ||\sigma_n - \sigma_m||.$$

Therefore λ_n is a Cauchy sequence and there exists $\text{s-}\lim_{n\to\infty} \lambda_n$.

(3) $\Rightarrow$ By the representation of $\lambda_J - i\,{}^*\lambda_J$ we have

$$\lambda_J - \lambda_n + i\,{}^*(\lambda_J - \lambda_n) = 2(a_{Jn} - 1)(\lambda_n - i\,{}^*\lambda_n) + 2\hat{\mu}_n^J,$$
$$4||\lambda_J - \lambda_n||^2 \ge 4(a_{Jn} - 1)^2||\varphi||^2 + 2(a_J + 2a_{Jn} - 1 - 2a_{Jn}{}^2)||\varphi||^2 \ge 0.$$

Since $\lim_{n\to\infty} ||\lambda_J - \lambda_n||^2 = 0$, we get $\lim_{n\to\infty} a_{Jn} = 1,\ a_J = 1,\ \tilde{X}(\varphi) = \{1\}$. □

Theorem 5.2 *The following three conditions are equivalent*

$$(1)\ X(\varphi)=\left\{\frac{1}{2}\right\} \Longleftrightarrow (2)\ \text{BW-}\lim_{n\to\infty}\lambda_n(\varphi)=\frac{\varphi}{2} \Longleftrightarrow (3)\ a_{Jn}=\frac{1}{2}\ \textit{for } J \textit{ admissible for } \varphi.$$

Proof.

(1) $\Rightarrow a_J(\varphi)=\frac{1}{2}$, then by Corollary 5.1 $\lambda_J=\frac{\varphi}{2}$. From Lemma 3.5, $\text{BW-}\lim_{n\to\infty}\lambda_n(\varphi)=\frac{\varphi}{2}$. And for J admissible for φ we have

$$\lambda_J-\lambda_n=\frac{\varphi}{2}-\lambda_n=(a_{Jn}-1)(\lambda_n-i\,{}^*\lambda_n)+\mu_n^J-i\,{}^*\mu_n^J=-\frac{\lambda_n-i\,{}^*\lambda_n}{2},$$

$$(a_{Jn}-\frac{1}{2})(\lambda_n-i\,{}^*\lambda_n)+\mu_n^J-i\,{}^*\mu_n^J=0.$$

It follows that $a_{Jn}=\frac{1}{2},\quad \mu_n^J=i\,{}^*\mu_n^J$.

(2) $\Rightarrow$ For any J admissible for φ, we have $\lambda_J=\frac{\varphi}{2}$ and $a_{Jn}=\frac{1}{2}$.

(3) $\Rightarrow a_{Jn}=\frac{1}{2}$, hence $a_J=\frac{1}{2}$ and $X(\varphi)=\left\{\frac{1}{2}\right\}$. □

For $\varphi\in\Lambda_a$, let $\lambda_J=a_{Jn}\lambda_n+(1-a_{Jn})i\,{}^*\lambda_n+\mu_n^J-i\,{}^*\mu_n^J$, and take a subsequence $K\subset J$ such that $\text{BW-}\lim_{K\ni k\to\infty}\mu_k^J=\mu_K^J$.

Proposition 5.2.

$$\mu_K^J-i\,{}^*\mu_K^J=(1-a_K)(\lambda_K-i\,{}^*\lambda_K).$$

We have a representation

$$\mu_K^J=A_n\mu_n^J+(A_n-2(1-a_J))i\,{}^*\mu_n^J+\frac{(1-a_J)(2a_{Jn}-1)}{2}(\lambda_n-i\,{}^*\lambda_n)+\sigma_n^K+i\,{}^*\sigma_n^K,$$

and

$$\|\mu_K^J\|^2=\frac{\|\varphi\|^2}{4}(3a_J-1-2a_J{}^2)\lim_{K\ni k\to\infty}A_k.$$

Proof.

$$\begin{aligned}\lambda_J-i\,{}^*\lambda_J&=(2a_{Jn}-1)(\lambda_n-i\,{}^*\lambda_n)+2(\mu_n^J-i\,{}^*\mu_n^J)\\&=(2a_K-1)(\lambda_K-i\,{}^*\lambda_K)+2(\mu_K^J-i\,{}^*\mu_K^J)=\lambda_K-i\,{}^*\lambda_K.\end{aligned}$$

Hence,

$$(1-a_K)(\lambda_K-i\,{}^*\lambda_K)=(\mu_K^J-i\,{}^*\mu_K^J).$$

Further set

$$\mu_K^J=A_n\mu_n^J+B_ni\,{}^*\mu_n^J+C_n\lambda_n+D_ni\,{}^*\lambda_n+\sigma_n^K+i\,{}^*\tau_n^K,$$

where $\mu_n^J\in\Lambda_n(\varphi)$, σ_n^K, $\tau_n^K\in\Lambda_n(\varphi,J)$(the orthogonal complement of $\{\mu_n^J\}$ in $\Lambda_n(\varphi)$). Then

$$\mu_K^J-i\,{}^*\mu_K^J=\text{BW-}\lim_{K\ni k\to\infty}\{\lambda_J-\lambda_K-(a_{Jk}-1)(\lambda_k-i\,{}^*\lambda_k)\}$$

$$= (1 - a_J)(\lambda_J - i\,{}^*\lambda_J) = (1 - a_J)(2a_{Jn} - 1)(\lambda_n - i\,{}^*\lambda_n) + 2(1 - a_J)(\mu_n^J - i\,{}^*\mu_n^J)$$
$$= (A_n - B_n)(\mu_n^J - i\,{}^*\mu_n^J) + (C_n - D_n)(\lambda_n - i\,{}^*\lambda_n) + (\sigma_n^K - \tau_n^K) + i\,{}^*(\tau_n^K - \sigma_n^K).$$

Hence we have

$$A_n - B_n = 2(1 - a_J),\ \ C_n - D_n = (1 - a_J)(2a_{Jn} - 1),\ \ \sigma_n^K - \tau_n^K = 0.$$

From

$$< \mu_K^J,\ \varphi >= \lim_{K \ni k \to \infty} < \mu_k^J,\ \varphi >= \lim_{K \ni k \to \infty} < \mu_k^J, \lambda_k + i\,{}^*\lambda_k >= 0,$$

we have

$$0 =< \mu_K^J,\ \lambda_n + i\,{}^*\lambda_n >= (C_n + D_n)||\lambda_n||^2.$$

It follows that $D_n = -C_n$. From the representation

$$\mu_K^J = A_n \mu_n^J + (A_n - 2(1 - a_J))i\,{}^*\mu_n^J + \frac{(1 - a_J)(2a_{Jn} - 1)}{2}(\lambda_n - i\,{}^*\lambda_n) + \sigma_n^K + i\,{}^*\sigma_n^K,$$

we have

$$||\mu_K^J||^2 = \lim_{K \ni k \to \infty} < \mu_k^J,\ \mu_K^J >= \lim_{K \ni k \to \infty} A_k ||\mu_k^J||^2$$
$$= \lim_{K \ni k \to \infty} A_k (a_J + 2a_{Jk} - 1 - 2{a_{Jk}}^2)\frac{||\varphi||^2}{4}$$
$$= \frac{||\varphi||^2}{4}(3a_J - 1 - 2{a_J}^2) \lim_{K \ni k \to \infty} A_k.$$

□

Corollary 5.2. *If* $\frac{1}{2} < a_J < 1$, *then* $1 \leq A_K + a_J \leq \frac{3}{2}$.

Proof. If $a_J \neq \frac{1}{2},\ 1$, by

$$A_K = \lim_{K \ni k \to \infty} A_k = \frac{-4||\mu_K^J||^2}{(2a_J - 1)(a_J - 1)||\varphi||^2}$$

and

$$||\mu_K^J||^2 = \{|A_n|^2 + |A_n - 2(1 - a_J)|^2\}||\mu_n^J||^2 + \frac{1}{4}(1 - a_J)^2(2a_{Jn} - 1)^2\ 2||\lambda_n||^2 + 2||\sigma_n^K||^2,$$

we have

$$\lim_{K \ni k \to \infty} ||\sigma_k^K||^2 = \frac{1}{2}\{\frac{||\varphi||^2}{4}(3a_J - 1 - 2{a_J}^2)A_K$$
$$-(2{A_K}^2 - 4(1 - a_J)A_K + 4(1 - a_J)^2)(3a_J - 1 - 2{a_J}^2)\frac{||\varphi||^2}{4} - \frac{1}{4}(1 - a_J)^2(2a_J - 1)^2||\varphi||^2\}.$$

Put in order

$$0 \leq \lim_{K \ni k \to \infty} ||\sigma_k^K||^2$$
$$= \frac{||\varphi||^2}{8}(2{a_J}^2 - 3a_J + 1)\{2{A_K}^2 + (4a_J - 5)A_K + 4(1 - a_J)^2 - (2{a_J}^2 - 3a_J + 1)\}$$

$$= \frac{||\varphi||^2}{8}(2a_J{}^2 - 3a_J + 1)(2A_K{}^2 + (4a_J - 5)A_K + 2a_J{}^2 - 5a_J + 3)$$
$$= \frac{||\varphi||^2}{8}(2a_J - 1)(a_J - 1)(2A_K + 2a_J - 3)(A_K + a_J - 1).$$

Therefore if $\frac{1}{2} < a_J < 1$, then $1 \leq A_K + a_J \leq \frac{3}{2}$. □

Let $\lambda_J - i\,{}^*\lambda_J = \nu_n + i\,{}^*\nu_n', \ \nu_n, \ \nu_n' \in \Lambda_n$. Since $\lambda_J - i\,{}^*\lambda_J = -i\,{}^*\nu_n - \nu_n'$, we have $\nu_n = -\nu_n'$. Hence

$$\lambda_J - i\,{}^*\lambda_J = \nu_n - i\,{}^*\nu_n, \quad ||\nu_n|| \leq 2||\lambda_J||.$$

From

$$\lambda_J - i\,{}^*\lambda_J = (2a_{Jn} - 1)(\lambda_n - i\,{}^*\lambda_n) + 2(\mu_n^J - i\,{}^*\mu_n^J) = \nu_n - i\,{}^*\nu_n,$$

it follows that

$$||\nu_n||^2 = (2a_{Jn} - 1)^2\frac{||\varphi||^2}{2} + 4||\mu_n^J||^2 = (2a_J - 1)\frac{||\varphi||^2}{2},$$

and

$$\nu_K = \text{BW-}\lim_{K \ni k \to \infty} \nu_k = (2a_J - 1)\lambda_J + 2\mu_K^J.$$

6. A sequence of canonical behavior spaces

We denote by $\Lambda_{cs}, \ \Lambda_{cbw}$ the spaces $\Lambda_s, \ \Lambda_{bw}$ associated with a sequence of canonical behavior spaces $\{\Lambda_{cm}\}$. Let $\underline{\Lambda}_{cs} = \{\lambda \in \Lambda_h :^{\exists} \lambda_n \in \underline{\Lambda}_{cn}$ such that $\text{s-}\lim_{n\to\infty} \lambda_n = \lambda\}$. Then we have

$$\underline{\Lambda}_{cn} \subset \underline{\Lambda}_{cs} \subset \Lambda_{cs} \subset \Lambda_{cbw}, \ \underline{\Lambda}_{cn} \subset \Lambda_{cn} \subset \Lambda_{cbw},$$
$$\underline{\Lambda}_{cn}^{\perp} \supset \underline{\Lambda}_{cs}^{\perp} \supset \Lambda_{cs}^{\perp} \supset \Lambda_{cbw}^{\perp}, \ \underline{\Lambda}_{cn}^{\perp} \supset \Lambda_{cn}^{\perp} = i\,{}^*\Lambda_{cn} \supset \Lambda_{cbw}^{\perp},$$
$$\Lambda_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp} \subset \underline{\Lambda}_{cn}^{\perp} \cap i\,{}^*\underline{\Lambda}_{cn}^{\perp}.$$

Since a differential $\omega \in \Lambda_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp}$ is orthogonal to $ie^{i\theta_j}\,{}^*\sigma_{A_j}, \ ie^{i\theta_j}\,{}^*\sigma_{B_j}$, the periods of ω along $A_j, \ B_j$ lie on L_j. Further $i\,{}^*\omega, \ \omega + i\,{}^*\omega, \ \omega - i\,{}^*\omega$ belong to $\Lambda_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp}$. Since $\Lambda_{he} \cap \Lambda_{ho} \subset \Lambda_{cs}, \ {}^*\Lambda_{he} \cap {}^*\Lambda_{ho} \subset i\,{}^*\Lambda_{cs}$, we have

$$\Lambda_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp} \subset \mathit{Closure\ of}\ \{\ \Lambda_{he} + \Lambda_{ho}\} \ \cap \ \mathit{Closure\ of}\ \{\ {}^*\Lambda_{he} + {}^*\Lambda_{ho}\}\ .$$

Here take an orthonormal basis $\{\varphi_k\}_{k=1}^N$ of the subspace which consists of holomorphic differentials in $\Lambda_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp}$ and an orthonormal basis $\{\psi_\ell\}_{\ell=1}^{N'}$ of the subspace which consists of anti-holomorphic differentials in $\Lambda_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp}$. Note that

$$< \varphi_k + \psi_k, i\,{}^*(\varphi_k + \psi_k) >=< \varphi_k + \psi_k, \varphi_k - \psi_k >= 0,$$
$$< \varphi_k + \psi_k, \varphi_\ell + \psi_\ell >=< \varphi_k + \psi_k, \varphi_\ell - \psi_\ell >= 0 \ \ (k \neq \ell).$$

Let N'' be the small one of N and N'. We denote by Λ_1 the subspace spanned by $\{\varphi_k + \psi_k\}_{k=1}^{N''}$, by Λ_2 the subspace spanned by$\{\varphi_k - \psi_k\}_{k=1}^{N''}$ and by Λ_3 the subspace spanned by remained differentials. We have

$$\Lambda_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp} = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \subset \mathit{Closure\ of}\{\Lambda_{he} + \Lambda_{ho}\}, \text{ and } i\,{}^*\Lambda_1 = \Lambda_2.$$

The periods of a differential in Λ_1 along A_j, B_j lie on L_j. It follows

Theorem 6.1. *When Λ_3 is empty, $\Lambda_\chi = \Lambda_{cs} + \Lambda_1$ satisfies*

$$(1)\ \Lambda_\chi = i\,{}^*\Lambda_\chi^\perp \subset Closure\ of\{\Lambda_{he} + \Lambda_{ho}\},$$

$$(2)\ \int_{A_j} \lambda \in L_i,\ \int_{B_j} \lambda \in L_i,\ for\ j \in \boldsymbol{J}_i,\ \lambda \in \Lambda_\chi.$$

Hence $\Lambda_{cs} + \Lambda_1$ is a behavior space for $\{L_j\}$.

Hereafter we assume that Λ_3 is non empty and consists of holomorphic differentials, and check up on the differentials in $\Lambda_{cs}^\perp \cap i\,{}^*\Lambda_{cs}^\perp \cap \Lambda_a$. For a $\varphi \in \Lambda_{cs}^\perp \cap i\,{}^*\Lambda_{cs}^\perp \cap \Lambda_a$, consider the orthogonal decomposition

$$\varphi = \lambda_n + i\,{}^*\lambda_n, \lambda_n \in \Lambda_{cn}.$$

Lemma 6.1.

(1) $\lambda_n \in \underline{\Lambda}_{cn}^\perp$,

(2) *For $m > n$, $\lambda_n = \alpha_{nm} - i\,{}^*\alpha_{nm} + \beta_m^n + i\,{}^*\gamma_m^n$,*

where $\alpha_{nm} \in \underline{\Lambda}_{cm}$, β_m^n, $\gamma_m^n \in \Lambda_{cm} \cap \underline{\Lambda}_{cm}^\perp$. And

$\alpha_{nm} + \beta_m^n = a_{mn}\lambda_m + \underline{\mu}_m^n$, $-\alpha_{nm} + \gamma_m^n = (1 - a_{mn})\lambda_m - \underline{\mu}_m^n$, $\beta_m^n + \gamma_m^n = \lambda_m$.

Proof. (1) $\lambda_n = \varphi - i\,{}^*\lambda_n \in \Lambda_{cs}^\perp + i\,{}^*\Lambda_{cn} \subset \underline{\Lambda}_{cn}^\perp$.

(2) It is clear $\underline{\Lambda}_{cn} \subset \underline{\Lambda}_{cm}$, and $\underline{\Lambda}_{cn}^\perp = \underline{\Lambda}_{cm}^\perp + \underline{\Lambda}_{cm} \cap \underline{\Lambda}_{cn}^\perp$, by $\Lambda_{cm} \supset \underline{\Lambda}_{cm}$, $\Lambda_{cm} = \underline{\Lambda}_{cm} + \underline{\Lambda}_{cm}^\perp \cap \Lambda_{cm}$. Hence

$$\Lambda_h = \underline{\Lambda}_{cm} + \underline{\Lambda}_{cm}^\perp \cap \Lambda_{cm} + i\,{}^*\underline{\Lambda}_{cm} + i\,{}^*\underline{\Lambda}_{cm}^\perp \cap i\,{}^*\Lambda_{cm}.$$

We can put

$$\lambda_n = \alpha_{nm} + i\,{}^*\alpha'_{nm} + \beta_m^n + i\,{}^*\gamma_m^n,\ \alpha_{nm}, \alpha'_{nm} \in \underline{\Lambda}_{cm},\ \beta_m^n,\ \gamma_m^n \in \Lambda_{cm} \cap \underline{\Lambda}_{cm}^\perp.$$

Then

$$\varphi = \lambda_n + i\,{}^*\lambda_n = (\alpha_{nm} + \alpha'_{nm}) + i\,{}^*(\alpha_{nm} + \alpha'_{nm}) + (\beta_m^n + \gamma_m^n) + i\,{}^*(\beta_m^n + \gamma_m^n).$$

Since $\varphi \in \Lambda_{cs}^\perp \subset \underline{\Lambda}_{cm}^\perp$, we get $\alpha_{nm} + \alpha'_{nm} = 0$. From $\varphi = \lambda_m + i\,{}^*\lambda_m$, we get $\beta_m^n + \gamma_m^n = \lambda_m$. We have

$$\lambda_n = a_{mn}\lambda_m + (1 - a_{mn})i\,{}^*\lambda_m + \underline{\mu}_m^n - i\,{}^*\underline{\mu}_m^n = \alpha_{nm} - i\,{}^*\alpha_{nm} + \beta_m^n + i\,{}^*\gamma_m^n$$

and $a_{mn}\lambda_m + \underline{\mu}_m^n = \alpha_{nm} + \beta_m^n$, $(1 - a_{mn})\lambda_m - \underline{\mu}_m^n = -\alpha_{nm} + \gamma_m^n$.

Now $||\alpha_{nm}|| \le ||\varphi||, ||\beta_m^n|| \le ||\varphi||, ||\gamma_m^n|| \le ||\varphi||$. Hence, by Lemma 3.1, there exists

$$J \subset \mathbf{N} \text{ such that } \operatorname{BW-}\lim_{J \ni j \to \infty} \alpha_{nj} = \alpha_{nJ},\ \operatorname{BW-}\lim_{J \ni j \to \infty} \beta_j^n = \beta_J^n,\ \operatorname{BW-}\lim_{J \ni j \to \infty} \gamma_j^n = \gamma_J^n.$$

Then $\operatorname{BW-}\lim_{J \ni j \to \infty} \lambda_j = \operatorname{BW-}\lim_{J \ni j \to \infty} (\beta_j^n + \gamma_j^n) = \beta_J^n + \gamma_J^n = \lambda_J$. And

$$\lambda_n = \alpha_{nJ} - i\,{}^*\alpha_{nJ} + \beta_J^n + i\,{}^*\gamma_J^n.$$

□

Lemma 6.2.

$$\alpha_{nJ} \in \underline{\Lambda}_{cs}, \quad \beta_J^n,\ \gamma_J^n \in \underline{\Lambda}_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp}.$$

Proof. Since $\alpha_{nj} \in \underline{\Lambda}_{cj} \subset \underline{\Lambda}_{cs}$, for $\sigma \in \underline{\Lambda}_{cs}^{\perp}$ by Lemma 3.4 we have

$$< \alpha_{nJ}, \sigma >= \lim_{J \ni j \to \infty} < \alpha_{nj}, \sigma >= 0.$$

Hence $\alpha_{nJ} \in \underline{\Lambda}_{cs}$. For $\sigma \in \underline{\Lambda}_{cs} + i\,{}^*\Lambda_{cs}$, take $\sigma_{1j} \in \underline{\Lambda}_{cj}$, $\sigma_{2j} \in \Lambda_{cj}$ such that $\text{s-}\lim_{J \ni j \to \infty} \sigma_{1j} + i\,{}^*\sigma_{2j} = \sigma$. Noting that $\beta_j^n,\ \gamma_j^n \in i\,{}^*\Lambda_{cj}^{\perp} \cap \underline{\Lambda}_{cj}^{\perp}$, by Lemma 3.4

$$< \beta_J^n, \sigma >= \lim_{J \ni j \to \infty} < \beta_j^n, \sigma_{1j} + i\,{}^*\sigma_{2j} >= 0.$$

Hence $\beta_J^n \in \underline{\Lambda}_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp}$. Similarly $\gamma_J^n \in \underline{\Lambda}_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp}$. □

Lemma 6.3.

$$\alpha_{nJ} = \alpha_{nm} + \delta_m + i\,{}^*\eta_m, \ \delta_m, \eta_m \in \Lambda_{cm} \cap \underline{\Lambda}_{cs}^{\perp},$$

$$\text{s-}\lim_{m \to \infty} \delta_m = \text{s-}\lim_{m \to \infty} \eta_m = 0\,.$$

Proof. For $\alpha_{nJ} \in \Lambda_{cs}$, by $\underline{\Lambda}_{cm} \subset \Lambda_{cs} = \underline{\Lambda}_{cm} \oplus \underline{\Lambda}_{cm}^{\perp} \cap \Lambda_{cs}$ take the orthogonal decomposition

$$\alpha_{nJ} = \alpha'_m + \Delta'_m, \alpha'_m \in \underline{\Lambda}_{cm}, \Delta'_m \in \underline{\Lambda}_{cm}^{\perp} \cap \Lambda_{cs}.$$

For a representation

$$\alpha_{nJ} = \lambda_n + i\,{}^*\alpha_{nJ} - \beta_J^n - i\,{}^*\gamma_J^n = \alpha_{nm} - i\,{}^*(\alpha_{nm} - \alpha_{nJ}) + (\beta_m^n - \beta_J^n) + i\,{}^*(\gamma_m^n - \gamma_J^n),$$

we have

$$i\,{}^*\alpha_{nm} \in i\,{}^*\underline{\Lambda}_{cm} \subset i\,{}^*\Lambda_{cm} = \Lambda_{cm}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp},$$

$$i\,{}^*\alpha_{nJ} \in i\,{}^*\Lambda_{cs} \cap i\,{}^*\Lambda_{cbw} = i\,{}^*\Lambda_{cs} \cap \Lambda_{cs}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp},$$

$$\beta_m^n \in \Lambda_{cm} \cap \underline{\Lambda}_{cm}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp}, \beta_J^n \in \underline{\Lambda}_{cs}^{\perp} \cap i\,{}^*\Lambda_{cs}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp},$$

$$i\,{}^*\gamma_m^n \in i\,{}^*\Lambda_{cm} \cap i\,{}^*\underline{\Lambda}_{cm}^{\perp} = \Lambda_{cm}^{\perp} \cap i\,{}^*\underline{\Lambda}_{cm}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp},\ i\,{}^*\gamma_J^n \in i\,{}^*\underline{\Lambda}_{cs}^{\perp} \cap \Lambda_{cs}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp}.$$

Hence $\alpha'_m = \alpha_{nm}$. For the orthogonal decomposition of $\Delta'_m \in \underline{\Lambda}_{cm}^{\perp} \cap \Lambda_{cs}$:

$$\Delta'_m = \delta_m + i\,{}^*\eta_m, \delta_m,\ \eta_m \in \Lambda_{cm},$$

from $i\,{}^*\eta_m \in i\,{}^*\Lambda_{cm} = \Lambda_{cm}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp}$, we have $\delta_m = \Delta'_m - i\,{}^*\eta_m \in \underline{\Lambda}_{cm}^{\perp} \cap \Lambda_{cm}$.
Since

$$\Delta'_m = \alpha_{nJ} - \alpha_{nm} \in \Lambda_{cbw} = i\,{}^*\Lambda_{cs}^{\perp},\ \ i\,{}^*\Delta'_m \in \Lambda_{cs}^{\perp} \subset \underline{\Lambda}_{cm}^{\perp},$$

we have

$$\eta_m = i\,{}^*\Delta'_m - i\,{}^*\delta_m \in \underline{\Lambda}_{cm}^{\perp} \cap \Lambda_{cm}.$$

By Lemma 3.2

$$\text{BW-}\lim_{J \ni j \to \infty} \alpha_{nj} = \alpha_{nJ}, \text{ and } ||\alpha_{nJ}||^2 \geq \lim_{J \ni j \to \infty} ||\alpha_{nj}||^2 \geq ||\alpha_{nJ}||^2.$$

Hence by Lemma 3.3 $\text{s-}\lim_{J\ni j\to\infty} \alpha_{nj} = \alpha_{nJ}$. It follows that

$$\text{s-}\lim_{J\ni j\to\infty} \delta_j = \text{s-}\lim_{J\ni j\to\infty} \eta_j = 0.$$

$\underline{\Lambda}_{cm}$ are monotonically increased and the norms $||\alpha_{nm}||$ are monotonically increased. As a result

$$\text{s-}\lim_{m\to\infty} \delta_m = \text{s-}\lim_{m\to\infty} \eta_m = 0. \qquad \square$$

Proposition 6.1.

$$(1)\ \ ^{\exists}n \in \mathbf{N}\ \ s.t.\ \ \alpha_{nJ} = 0 \Longrightarrow\ \ \varphi = 0,$$

$$(2)\ \ ^{\exists}n \in \mathbf{N}\ \ s.t.\ \ a_{Jn} = 1 \Longrightarrow\ \ \varphi = 0.$$

Proof. (1) Let $\lambda_n = \beta_J^n + i\,^*\gamma_J^n \in \underline{\Lambda}_{cs}^{\perp} \cap i\,^*\Lambda_{cs}^{\perp} + i\,^*\underline{\Lambda}_{cs}^{\perp} \cap \Lambda_{cs}^{\perp} \subset \underline{\Lambda}_{cs}^{\perp} \cap i\,^*\underline{\Lambda}_{cs}^{\perp}$.
For $e^{i\theta_j}\sigma_{A_j},\ e^{i\theta_j}\sigma_{B_j} \in \underline{\Lambda}_{cm} \subset \underline{\Lambda}_{cs}$,

$$\Re(-ie^{-i\theta_j}\int_{A_j} \lambda_n) =< \lambda_n, ie^{i\theta_j}\,^*\sigma_{A_j} >= 0,$$

$$\Re(-ie^{-i\theta_j}\int_{B_j} \lambda_n) =< \lambda_n, ie^{i\theta_j}\,^*\sigma_{B_j} >= 0.$$

Hence $\int_{A_j} \lambda_n,\ \int_{B_j} \lambda_n \in L_j \cap L_{n\infty} = \{0\}\ for A_j,\ B_j \subset R - R_n$. Take a differential $\sigma \in S_n(L)$ such that $\lambda_n - \sigma \in \Lambda_{he}$. We have

$$\lambda_n - \sigma \in \Lambda_{he} \cap \{Closure\ of\{\ \Gamma_{he} + \Gamma_{ho}\}\ + i\Gamma_{ho}\} = \Gamma_{he} + i(\Gamma_{he} \cap \Gamma_{ho}).$$

Then

$$\lambda_n \in Closure\ of\{(\Gamma_{he} + \Gamma_{ho}) + i(\Gamma_{he} \cap \Gamma_{ho}) + S_n(L)\} = \underline{\Lambda}_{cn},$$

by Lemma 6.1 $\lambda_n = 0$, and $\varphi = 0$.

(2) Let $\lambda_J = a_{Jn}\lambda_n + (1 - a_{Jn})i\,^*\lambda_n + \hat{\mu}_n^J$. By Lemma 3.1 and Lemma 5.1 we have

$$\overline{\lim_{J\ni j\to\infty}} ||\mu_n^j||^2 = \lim_{J\ni j\to\infty} \frac{a_{jn}(1 - a_{jn})}{2}||\varphi||^2 = 0.$$

Hence $\mu_n^J = \text{BW-}\lim_{J\ni j\to\infty} \mu_n^j = 0$, and $\hat{\mu}_n^J = 0$. By Proposition 4.1 $\lambda_n = \lambda_J \in \Lambda_{cbw} = i\,^*\Lambda_{cs}^{\perp}$. From the assumption of φ, we have $i\,^*\lambda_n = \varphi - \lambda_n \in i\,^*\Lambda_{cs}^{\perp}$ and $\lambda_n \in \Lambda_{cs}^{\perp}$. We get $\alpha_{nj} = 0, \alpha_{nJ} = 0$. From (1) it follows that $\lambda_n = 0$. $\square$

Further consider the following orthogonal decomposition:

$$\alpha_{nJ} = \alpha_{nJm} + i\,^*\tilde{\alpha}_{nJm},\ \beta_J^n = \beta_{Jm}^n + i\,^*\tilde{\beta}_{Jm}^n,\ \gamma_J^n = \gamma_{Jm}^n + i\,^*\tilde{\gamma}_{Jm}^n,\ \underline{\mu}_J^n = \underline{\mu}_{Jm}^n + i\,^*\underline{\tilde{\mu}}_{Jm}^n,$$

where

$$\alpha_{nJm},\ \tilde{\alpha}_{nJm},\ \beta_{Jm}^n,\ \tilde{\beta}_{Jm}^n,\ \gamma_{Jm}^n,\ \tilde{\gamma}_{Jm}^n,\ \underline{\mu}_{Jm}^n,\ \underline{\tilde{\mu}}_{Jm}^n \in \Lambda_{cm}.$$

Lemma 6.4.

$$(1)\ \beta_m^n = \beta_{Jm}^n + \tilde{\gamma}_{Jm}^n + \delta_m - \eta_m,$$

$$(2)\ \gamma_m^n = \gamma_{Jm}^n + \tilde{\beta}_{Jm}^n - \delta_m + \eta_m,$$

$$(3)\ \underline{\mu}_{Jm}^n = -(a_{Jn} - 1)\beta_{Jm}^n - a_{Jn}\gamma_{Jm}^n + (\alpha_{nm} + \delta_m),$$

$$(4)\ \underline{\tilde{\mu}}_{Jm}^n = -(a_{Jn} - 1)\tilde{\beta}_{Jm}^n - a_{Jn}\tilde{\gamma}_{Jm}^n + \eta_m.$$

Proof. By Lemma 6.1, we have

$$\lambda_n = \alpha_{nJ} - i\,{}^*\alpha_{nJ} + \beta_J^n + i^*\gamma_J^n = \alpha_{nm} - i\,{}^*\alpha_{nm} + \beta_m^n + i\,{}^*\gamma_m^n$$

$$= \alpha_{nm} + \delta_m + i\,{}^*\eta_m - i\,{}^*\alpha_{nm} - i\,{}^*\delta_m - \eta_m + \beta_{Jm}^n + \tilde{\gamma}_{Jm}^n + i\,{}^*\gamma_{Jm}^n + i\,{}^*\tilde{\beta}_{Jm}^n$$

$$= \alpha_{nm} - i\,{}^*\alpha_{nm} + (\beta_{Jm}^n + \tilde{\gamma}_{Jm}^n + \delta_m - \eta_m) + i\,{}^*(\tilde{\beta}_{Jm}^n + \gamma_{Jm}^n - \delta_m + \eta_m).$$

Since

$$\beta_m^n,\ \gamma_m^n \in \Lambda_{cm} \cap \underline{\Lambda}_{cs}^{\perp},\ (\beta_{Jm}^n + \tilde{\gamma}_{Jm}^n + \delta_m - \eta_m),\ (\tilde{\beta}_{Jm}^n + \gamma_{Jm}^n - \delta_m + \eta_m) \in \Lambda_{cm},$$

we have

$$\beta_m^n = \beta_{Jm}^n + \tilde{\gamma}_{Jm}^n + \delta_m - \eta_m,\ \gamma_m^n = \tilde{\beta}_{Jm}^n + \gamma_{Jm}^n - \delta_m + \eta_m.$$

From

$$\alpha_{nJ} + \beta_J^n = a_{Jn}\lambda_J + \underline{\mu}_J^n,\ -\alpha_{nJ} + \gamma_J^n = (1 - a_{Jn})\lambda_J - \underline{\mu}_J^n,$$

we get

$$\underline{\mu}_{Jm}^n + i\,{}^*\underline{\tilde{\mu}}_{Jm}^n = \underline{\mu}_J^n = -(a_{Jn} - \tfrac{1}{2})\lambda_J + \alpha_{nJ} + \tfrac{1}{2}(\beta_J^n - \gamma_J^n)$$

$$= -a_{Jn}(\beta_J^n + \gamma_J^n) + \alpha_{nJ} + \beta_J^n$$

$$= -a_{Jn}(\beta_{Jm}^n + i\,{}^*\tilde{\beta}_{Jm}^n + \gamma_{Jm}^n + i\,{}^*\tilde{\gamma}_{Jm}^n) + (\alpha_{nm} + \delta_m + i\,{}^*\eta_m) + \beta_{Jm}^n + i\,{}^*\tilde{\beta}_{Jm}^n$$

$$= -(a_{Jn} - 1)\beta_{Jm}^n - a_{Jn}\gamma_{Jm}^n + (\alpha_{nm} + \delta_m) - i\,{}^*\{(a_{Jn} - 1)\tilde{\beta}_{Jm}^n + a_{Jn}\tilde{\gamma}_{Jm}^n - \eta_m\}.$$

Therefore

$$\underline{\mu}_{Jm}^n = -(a_{Jn} - 1)\beta_{Jm}^n - a_{Jn}\gamma_{Jm}^n + (\alpha_{nm} + \delta_m),$$

$$\underline{\tilde{\mu}}_{Jm}^n = -(a_{Jn} - 1)\tilde{\beta}_{Jm}^n - a_{Jn}\tilde{\gamma}_{Jm}^n + \eta_m.$$

□

Lemma 6.5.

$$(1)\ \beta_{Jm}^n + \gamma_{Jm}^n = a_{Jm}\lambda_m + \mu_m^J,$$

$$(2)\ \tilde{\beta}_{Jm}^n + \tilde{\gamma}_{Jm}^n = (1 - a_{Jm})\lambda_m - \mu_m^J,$$

$$(3)\ \lambda_m = \beta_{Jm}^n + \gamma_{Jm}^n + \tilde{\beta}_{Jm}^n + \tilde{\gamma}_{Jm}^n,$$

$$(4)\ \mu_m^J = (1 - a_{Jm})(\beta_{Jm}^n + \gamma_{Jm}^n) - a_{Jm}(\tilde{\beta}_{Jm}^n + \tilde{\gamma}_{Jm}^n).$$

Proof. By the representation

$$\lambda_J = \beta_J^n + \gamma_J^n = \beta_{Jm}^n + i\,{}^*\tilde{\beta}_{Jm}^n + \gamma_{Jm}^n + i\,{}^*\tilde{\gamma}_{Jm}^n$$

$$= a_{Jm}\lambda_m + (1 - a_{Jm})i\,{}^*\lambda_m + \mu_m^J - i\,{}^*\mu_m^J,$$

we have

$$\beta^n_{Jm} + \gamma^n_{Jm} = a_{Jm}\lambda_m + \mu^J_m,\ \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm} = (1 - a_{Jm})\lambda_m - \mu^J_m.$$

It follows that

$$\lambda_m = \beta^n_{Jm} + \gamma^n_{Jm} + \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm},$$

$$\mu^J_m = \frac{1}{2}(\beta^n_{Jm} + \gamma^n_{Jm} - \tilde{\beta}^n_{Jm} - \tilde{\gamma}^n_{Jm}) + (\frac{1}{2} - a_{Jm})\lambda_m$$

$$= \beta^n_{Jm} + \gamma^n_{Jm} - a_{Jm}(\beta^n_{Jm} + \gamma^n_{Jm} + \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm})$$

$$= (1 - a_{Jm})(\beta^n_{Jm} + \gamma^n_{Jm}) - a_{Jm}(\tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm}).$$ □

Lemma 6.6.

$$(1)\ ||\beta^n_{Jm} + \gamma^n_{Jm}||^2 = (a_J + 2a_{Jm} - 1)\frac{||\varphi||^2}{4},$$

$$(2)\ ||\tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm}||^2 = (a_J - 2a_{Jm} + 1)\frac{||\varphi||^2}{4},$$

$$(3) < \beta^n_{Jm} + \gamma^n_{Jm}, \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm} >= (1 - a_J)\frac{||\varphi||^2}{4}.$$

Proof. By Lemma 6.5

$$(1)\ ||\beta^n_{Jm} + \gamma^n_{Jm}||^2 = ||a_{Jm}\lambda_m + \mu^J_m||^2$$

$$= (2a_{Jm}{}^2 + a_J + 2a_{Jm} - 1 - 2a_{Jm}{}^2)\frac{||\varphi||^2}{4} = (a_J + 2a_{Jm} - 1)\frac{||\varphi||^2}{4},$$

$$(2)\ ||\tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm}||^2 = ||(1 - a_{Jm})\lambda_m - \mu^J_m||^2$$

$$= \{2(1 - a_{Jm})^2 + a_J + 2a_{Jm} - 1 - 2a_{Jm}{}^2\}\frac{||\varphi||^2}{4} = (a_J - 2a_{Jm} + 1)\frac{||\varphi||^2}{4},$$

$$(3) < \beta^n_{Jm} + \gamma^n_{Jm}, \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm} >=< a_{Jm}\lambda_m + \mu^J_m, (1 - a_{Jm})\lambda_m - \mu^J_m >$$

$$= \{2a_{Jm}(1 - a_{Jm}) - a_J - 2a_{Jm} + 1 + 2a_{Jm}{}^2\}\frac{||\varphi||^2}{4} = (1 - a_J)\frac{||\varphi||^2}{4}.$$ □

Lemma 6.7.

$$(1)\ \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \tilde{\gamma}^n_{Jj} > -||\tilde{\beta}^n_{Jj}||^2) = 0,$$

$$(2)\ \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \tilde{\beta}^n_{Jj} > - < \tilde{\beta}^n_{Jj}, \tilde{\gamma}^n_{Jj} >) = 0,$$

$$(3)\ \lim_{J\ni j\to\infty}(< \gamma^n_{Jj}, \tilde{\beta}^n_{Jj} > -||\tilde{\gamma}^n_{Jj}||^2) = 0,$$

$$(4)\ \lim_{J\ni j\to\infty}(< \gamma^n_{Jj}, \tilde{\gamma}^n_{Jj} > - < \tilde{\beta}^n_{Jj}, \tilde{\gamma}^n_{Jj} >) = 0,$$

(5) $\lim_{J\ni j\to\infty}(<\beta^n_{Jj},\gamma^n_{Jj}>+3<\tilde{\gamma}^n_{Jj},\tilde{\beta}^n_{Jj}>-||\alpha_{nj}||^2)=0,$

(6) $\lim_{J\ni j\to\infty}(<\beta^n_{Jj},\beta^n_{Jj}+\gamma^n_{Jj}>+<\tilde{\beta}^n_{Jj},\tilde{\beta}^n_{Jj}+\tilde{\gamma}^n_{Jj}>)=\frac{1}{4}(2a_{Jn}+a_J-1)||\varphi||^2.$

Proof. (1) By Lemma 6.4 we have

$$<\beta^n_J,\beta^n_m>=<\beta^n_{Jm}+i^*\tilde{\beta}^n_{Jm},\beta^n_{Jm}+\tilde{\gamma}^n_{Jm}+\delta_m-\eta_m>$$
$$=<\beta^n_{Jm},\beta^n_{Jm}+\tilde{\gamma}^n_{Jm}>+<\beta^n_{Jm},\delta_m-\eta_m>.$$

Using Lemma 3.4 we have

$$||\beta^n_{Jj}||^2+||\tilde{\beta}^n_{Jj}||^2=||\beta^n_J||^2=\lim_{J\ni j\to\infty}<\beta^n_J,\beta^n_j>=\lim_{J\ni j\to\infty}(||\beta^n_{Jj}||^2+<\beta^n_{Jj},\tilde{\gamma}^n_{Jj}>),$$

$$\therefore \lim_{J\ni j\to\infty}(<\beta^n_{Jj},\tilde{\gamma}^n_{Jj}>-||\tilde{\beta}^n_{Jj}||^2)=0.$$

Similarly we have the following:

$$(2)\ <\beta^n_J,i^*\beta^n_m>=<\beta^n_{Jm}+i^*\tilde{\beta}^n_{Jm},i^*\beta^n_{Jm}+i^*\tilde{\gamma}^n_{Jm}+i^*\delta_m-i^*\eta_m>$$

$$=<\tilde{\beta}^n_{Jm},\beta^n_{Jm}+\tilde{\gamma}^n_{Jm}>+<\tilde{\beta}^n_{Jm},\delta_k-\eta_m>,$$

$$<\beta^n_J,i^*\beta^n_J>=<\beta^n_{Jj}+i^*\tilde{\beta}^n_{Jj},i^*\beta^n_J+\tilde{\beta}^n_{Jj}>=2<\beta^n_{Jj},\tilde{\beta}^n_{Jj}>$$

$$=\lim_{J\ni j\to\infty}<\beta^n_J,i^*\beta^n_j>=\lim_{J\ni j\to\infty}(<\tilde{\beta}^n_{Jj},\beta^n_{Jj}>+<\tilde{\beta}^n_{Jj},\tilde{\gamma}^n_{Jj}>),$$

$$\therefore \lim_{J\ni j\to\infty}(<\beta^n_{Jj},\tilde{\beta}^n_{Jj}>-<\tilde{\beta}^n_{Jj},\tilde{\gamma}^n_{Jj}>)=0,$$

$$(3)\ <\gamma^n_J,\gamma^n_m>=<\gamma^n_{Jm}+i^*\tilde{\gamma}^n_{Jm},\tilde{\beta}^n_{Jm}+\gamma^n_{Jm}-\delta_m+\eta_m>$$

$$=<\gamma^n_{Jm},\tilde{\beta}^n_{Jm}+\gamma^n_{Jm}>+<\gamma^n_{Jm},-\delta_m+\eta_m>,$$

$$||\gamma^n_{Jj}||^2+||\tilde{\gamma}^n_{Jj}||^2=||\gamma^n_J||^2=\lim_{J\ni j\to\infty}<\gamma^n_J,\gamma^n_j>=\lim_{J\ni j\to\infty}(||\gamma^n_{Jj}||^2+<\gamma^n_{Jj},\tilde{\beta}^n_{Jj}>),$$

$$\therefore \lim_{J\ni j\to\infty}(<\gamma^n_{Jj},\tilde{\beta}^n_{Jj}>-||\tilde{\gamma}^n_{Jj}||^2)=0,$$

$$(4)\ <\gamma^n_J,i^*\gamma^n_m>=<\gamma^n_{Jm}+i^*\tilde{\gamma}^n_{Jm},i^*\tilde{\beta}^n_{Jm}+i^*\gamma^n_{Jm}-i^*\delta_m+i^*\eta_m>$$

$$=< \tilde{\gamma}^n_{Jm}, \gamma^n_{Jm} + \tilde{\beta}^n_{Jm} > + < \tilde{\gamma}^n_{Jm}, -\delta_m + \eta_m >,$$

$$< \gamma^n_J, i\,^*\gamma^n_J >=< \gamma^n_{Jj} + i\,^*\tilde{\gamma}^n_{Jj}, i\,^*\gamma^n_J + \tilde{\gamma}^n_{Jj} >= 2 < \gamma^n_{Jj}, \tilde{\gamma}^n_{Jj} >$$

$$= \lim_{J\ni j\to\infty} < \gamma^n_j, \gamma^n_j >= \lim_{J\ni j\to\infty}(< \tilde{\gamma}^n_{Jj}, \gamma^n_{Jj} > + < \tilde{\gamma}^n_{Jj}, \tilde{\beta}^n_{Jj} >),$$

$$\therefore \lim_{J\ni j\to\infty}(< \gamma^n_{Jj}, \tilde{\gamma}^n_{Jj} > - < \tilde{\beta}^n_{Jj}, \tilde{\gamma}^n_{Jj} >) = 0.$$

(5) It follows that

$$||\alpha_{nm} - i\,^*\alpha_{nm} + \beta^n_m + i\,^*\gamma^n_m||^2 = ||\lambda_n||^2 = ||\lambda_m||^2 = ||\beta^n_m + \gamma^n_m||^2,$$

$$||\alpha_{nm}||^2 =< \beta^n_m, \gamma^n_m >=< \beta^n_{Jm} + \tilde{\gamma}^n_{Jm} + \delta_m - \eta_m, \tilde{\beta}^n_{Jm} + \gamma^n_{Jm} - \delta_m + \eta_m >,$$

$$\lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \tilde{\beta}^n_{Jj} > + < \beta^n_{Jj}, \gamma^n_{Jj} > + < \tilde{\gamma}^n_{Jj}, \tilde{\beta}^n_{Jj} > + < \tilde{\gamma}^n_{Jj}, \gamma^n_{Jj} > -||\alpha_{nj}||^2) = 0,$$

$$\therefore \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \gamma^n_{Jj} > +3 < \tilde{\gamma}^n_{Jj}, \tilde{\beta}^n_{Jj} > -||\alpha_{nj}||^2) = 0.$$

(6) By Lemma 6.1 we have

$$< \beta^n_m, \lambda_m >=< \alpha_{nm} + \beta^n_m, \lambda_m >=< a_{mn}\lambda_m + \underline{\mu}^n_m, \lambda_m >= \frac{a_{mn}}{2}||\varphi||^2$$

$$=< \beta^n_{Jm} + \tilde{\gamma}^n_{Jm} + \delta_m - \eta_m, \beta^n_{Jm} + \tilde{\beta}^n_{Jm} + \gamma^n_{Jm} + \tilde{\gamma}^n_{Jm} >,$$

$$\frac{a_{Jn}}{2}||\varphi||^2 = \lim_{J\ni j\to\infty}\frac{a_{jn}}{2}||\varphi||^2 = \lim_{J\ni j\to\infty} < \beta^n_{Jj} + \tilde{\gamma}^n_{Jj}, \beta^n_{Jj} + \tilde{\beta}^n_{Jj} + \gamma^n_{Jj} + \tilde{\gamma}^n_{Jj} >$$

$$= \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} > + < \beta^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} > + < \tilde{\gamma}^n_{Jj}, \beta^n_{Jj} + \tilde{\gamma}^n_{Jj} > + < \tilde{\gamma}^n_{Jj}, \tilde{\beta}^n_{Jj} + \gamma^n_{Jj} >)$$

$$= \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} > + < \tilde{\beta}^n_{Jj}, \tilde{\gamma}^n_{Jj} > +2||\tilde{\beta}^n_{Jj}||^2 + ||\tilde{\gamma}^n_{Jj}||^2 + 2 < \tilde{\gamma}^n_{Jj}, \tilde{\beta}^n_{Jj} >)$$

$$= \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} > + < \tilde{\beta}^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} > +||\tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj}||^2),$$

$$\therefore \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} > + < \tilde{\beta}^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >) = \lim_{J\ni j\to\infty}\frac{1}{4}(2a_{Jn} - a_J + 2a_{Jj} - 1)||\varphi||^2$$

$$= \frac{1}{4}(2a_{Jn} + a_J - 1)||\varphi||^2.$$

□

From Lemma 6.5 we can put

$$\beta^n_{Jm} = x_m\lambda_m + y_m\mu^J_m + \omega_m,$$

$$\gamma^n_{Jm} = (a_{Jm} - x_m)\lambda_m + (1 - y_m)\mu^J_m - \omega_m,$$

$$\tilde{\beta}^n_{Jm} = z_m\lambda_m - w_m\mu^J_m + \tilde{\omega}_m,$$

$$\tilde{\gamma}^n_{Jm} = (1 - a_{Jm} - z_m)\lambda_m + (w_m - 1)\mu^J_m - \tilde{\omega}_m.$$

Lemma 6.8.

$$(1)\ x_J = \lim_{J\ni j\to\infty} x_j = \frac{1}{2}(2a_{Jn} + a_J - 1),$$

$$(2)\ z_J = \lim_{J\ni j\to\infty} z_j = \frac{1}{2}(-a_J + 1).$$

Proof. (1) We have

$$\lim_{J\ni j\to\infty} x_j\frac{||\varphi||^2}{2} = \lim_{J\ni j\to\infty} < x_j\lambda_j + y_j\mu^J_j + \omega_j, \lambda_j >= \lim_{J\ni j\to\infty} < \beta^n_{Jj}, \lambda_j >$$

$$= \lim_{J\ni j\to\infty} < \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} + \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >$$

$$= \lim_{J\ni j\to\infty} (< \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} > + < \tilde{\beta}^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >)$$

$$= \frac{1}{4}(2a_{Jn} + a_J - 1)||\varphi||^2.$$

Hence

$$x_J = \lim_{J\ni j\to\infty} x_j = \frac{1}{2}(2a_{Jn} + a_J - 1).$$

(2) We have

$$\lim_{J\ni j\to\infty} z_j\frac{||\varphi||^2}{2} = \lim_{J\ni j\to\infty} < z_j\lambda_j - w_j\mu^J_m + \tilde{\omega}_j, \lambda_j >= \lim_{J\ni j\to\infty} < \tilde{\beta}^n_{Jj}, \lambda_j >$$

$$= \lim_{J\ni j\to\infty} < \tilde{\beta}^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} + \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >$$

$$= \lim_{J\ni j\to\infty} (< \tilde{\beta}^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} > + < \tilde{\beta}^n_{Jm}, \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm} >)$$

$$= \lim_{J \ni j \to \infty} (< \tilde{\gamma}^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} > + < \tilde{\beta}^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >)$$

$$= \frac{1}{4}(-a_J + 1)\|\varphi\|^2 \text{ (by Lemma 6.6).}$$

Therefore

$$z_J = \lim_{J \ni j \to \infty} z_j = \tfrac{1}{2}(-a_J + 1). \qquad \square$$

Lemma 6.9. *When* $\frac{1}{2} < a_J < 1$,

$$\lim_{J \ni j \to \infty} (y_j + w_j) = 2.$$

Proof. By $\underline{\mu}^n_m$, $\mu^J_m \in \Lambda_m(\varphi)$ and their representation we have

$$< \underline{\mu}^n_m, \mu^J_m > = \frac{1}{2} < \underline{\mu}^n_m - i\,{}^*\underline{\mu}^n_m, \mu^J_m - i\,{}^*\mu^J_m >$$

$$= \frac{1}{8} < \lambda_n - i\,{}^*\lambda_n - (2a_{mn} - 1)(\lambda_m - i\,{}^*\lambda_m), \lambda_J - i\,{}^*\lambda_J - (2a_{Jm} - 1)(\lambda_m - i\,{}^*\lambda_m) >$$

$$= \frac{1}{8}(< \lambda_n - i\,{}^*\lambda_n, \lambda_J - i\,{}^*\lambda_J > + (2a_{mn} - 1)(2a_{Jm} - 1)\|\varphi\|^2$$

$$- < (2a_{mn} - 1)(\lambda_J - i\,{}^*\lambda_J) + (2a_{Jm} - 1)(\lambda_n - i\,{}^*\lambda_n), (\lambda_m - i\,{}^*\lambda_m) >).$$

And by Lemma 5.1

$$< \underline{\mu}^n_m, \mu^J_m > = \frac{1}{8}\{(2a_{Jn} - 1)\|\varphi\|^2 - (2a_{mn} - 1)(2a_{Jm} - 1)\|\varphi\|^2 + (2a_{Jm} - 1)(2a_{mn} - 1)\|\varphi\|^2$$

$$+ (2a_{mn} - 1)(2a_{Jm} - 1)\|\varphi\|^2\}$$

$$= \frac{1}{8}\{(2a_{Jn} - 1) - (2a_{mn} - 1)(2a_{Jm} - 1)\}\ \|\varphi\|^2.$$

Therefore

$$\lim_{J \ni j \to \infty} < \underline{\mu}^n_j, \mu^J_j > = \frac{1}{4}(2a_{Jn} - 1)(1 - a_J)\|\varphi\|^2.$$

On the other hand, by Lemma 6.1 and 6.4,

$$\alpha_{nm} - a_{mn}\lambda_m - \underline{\mu}^n_m = -\beta^n_m = -\beta^n_{Jm} - \tilde{\gamma}^n_{Jm} - \delta_m + \eta_m$$

$$= -x_m\lambda_m - y_m\mu^J_m - \omega_m - (1 - a_{Jm} - z_m)\lambda_m - (w_m - 1)\mu^J_m + \tilde{\omega}_m - \delta_m + \eta_m$$

$$= (z_m - x_m + a_{Jm} - 1)\lambda_m + (1 - y_m - w_m)\mu^J_m - \omega_m + \tilde{\omega}_m - \delta_m + \eta_m.$$

We get

$$< \underline{\mu}_m^n, \mu_m^J >=< \alpha_{nm} - (z_m - x_m + a_{nm} + a_{Jm} - 1)\lambda_m - (1 - y_m - w_m)\mu_m^J + \omega_m - \tilde{\omega}_m + \delta_m - \eta_m, \mu_m^J >$$

$$= (y_m + w_m - 1)\|\mu_m^J\|^2 + < \alpha_{nm} + \delta_m - \eta_m, \mu_m^J > .$$

By Lemma 5.2 we have

$$\|\mu_m^J\|^2 = (a_J + 2a_{Jm} - 1 - 2{a_{Jm}}^2)\frac{\|\varphi\|^2}{4}.$$

From $\alpha_{nm} \in \underline{\Lambda}_{cm}$, $\mu_m^J \in \Lambda_{cm}$, we get

$$< \alpha_{nm}, \mu_m^J >=< \alpha_{nm}, \mu_m^J - i\,{}^*\mu_m^J >$$
$$= \frac{1}{2} < \alpha_{nm}, \lambda_J - i\,{}^*\lambda_J - (2a_{Jm} - 1)(\lambda_m - i\,{}^*\lambda_m) > .$$

Since $\varphi \in \Lambda_{cs}^\perp \subset \underline{\Lambda}_{cm}^\perp$, $i\,{}^*\lambda_m \in i\,{}^*\Lambda_{cm} = \Lambda_{cm}^\perp \subset \underline{\Lambda}_{cm}^\perp$, we get $\lambda_m = \varphi - i\,{}^*\lambda_m \in \underline{\Lambda}_{cm}^\perp$.

By Lemma 6.2 $\lambda_J \in \underline{\Lambda}_{cs}^\perp \cap i\,{}^*\Lambda_{cs}^\perp \subset \underline{\Lambda}_{cm}^\perp$, i.e. $i\,{}^*\lambda_J \in i\,{}^*\underline{\Lambda}_{cs}^\perp \cap \Lambda_{cs}^\perp \subset \underline{\Lambda}_{cm}^\perp$. Hence $< \alpha_{nm}, \mu_m^J >= 0$. It follows that

$$\frac{1}{4}(2a_{Jn} - 1)(1 - a_J)\|\varphi\|^2 = \lim_{J \ni j \to \infty} < \underline{\mu}_j^n, \mu_j^J >$$

$$= \lim_{J \ni j \to \infty} (y_j + w_j - 1)(3a_J - 1 - 2{a_J}^2)\frac{\|\varphi\|^2}{4}$$

$$= \frac{1}{4} \lim_{J \ni j \to \infty} (y_j + w_j - 1)(2a_J - 1)(1 - a_J)\|\varphi\|^2.$$

Noting that $(2a_J - 1)(1 - a_J) \neq 0$, we get

$$\lim_{J \ni j \to \infty}(y_j + w_j - 1) = 1 \text{ i.e., } \lim_{J \ni j \to \infty}(y_j + w_j) = 2. \qquad \square$$

When $(2a_J - 1)(1 - a_J) \neq 0$, for sufficiently large m, we may suppose that

$$\|\mu_m^J\|^2 = (a_J + 2a_{Jm} - 1 - 2{a_{Jm}}^2)\frac{\|\varphi\|^2}{4} \geq (2a_J - 1)(1 - a_J)\frac{\|\varphi\|^2}{8}.$$

Since $\{y_m\}$, $\{w_m\}$ is bounded. By Lemma 3.1 there exists $K \subset J$ such that

$$\lim_{K \ni k \to \infty} y_k = y_K, \quad \lim_{K \ni k \to \infty} w_k = w_K,$$

$$\text{BW-}\lim_{K \ni k \to \infty} \underline{\mu}_k^n = \underline{\mu}_K^n, \quad \text{BW-}\lim_{K \ni k \to \infty} \omega_k = \omega_K, \quad \text{BW-}\lim_{K \ni k \to \infty} \tilde{\omega}_k = \tilde{\omega}_K.$$

Then

$$\alpha_{nJ} - a_{Jn}\lambda_J - \underline{\mu}_K^n = \text{BW-}\lim_{K \ni k \to \infty} (\alpha_{nk} - a_{kn}\lambda_k - \underline{\mu}_k^n)$$

$$= (\frac{1}{2}(-a_J + 1) - \frac{1}{2}(2a_{Jn} + a_J - 1) + a_J - 1)\lambda_J + (1 - y_K - w_K)\mu_K^J - \omega_K + \tilde{\omega}_K$$
$$= -a_{Jn}\lambda_J - \mu_K^J - \omega_K + \tilde{\omega}_K.$$

Hence we have

$$\alpha_{nJ} - \underline{\mu}_K^n = -\mu_K^J - \omega_K + \tilde{\omega}_K.$$

On the other hand, by the representation of λ_J and Lemma 6.3

$$\begin{aligned}
\alpha_{nJ} - \underline{\mu}_K^n &= a_{Jn}\lambda_J - \beta_J^n \\
&= a_{Jn}(a_{Jm}\lambda_m + (1 - a_{Jm})i^*\lambda_m + \mu_m^J - i^*\mu_m^J) - \beta_{Jm}^n - i^*\tilde{\beta}_{Jm}^n \\
&= a_{Jn}(a_{Jm}\lambda_m + (1 - a_{Jm})i^*\lambda_m + \mu_m^J - i^*\mu_m^J) \\
&\quad -x_m\lambda_m - y_m\mu_m^J - \omega_m - z_m i^*\lambda_m + w_m i^*\mu_m^J - i^*\tilde{\omega}_m \\
&= (a_{Jn}a_{Jm} - x_m)\lambda_m + (a_{Jn} - a_{Jn}a_{Jm} - z_m)i^*\lambda_m \\
&\quad +(a_{Jn} - y_m)\mu_m^J - (a_{Jn} - w_m)i^*\mu_m^J - \omega_m - i^*\tilde{\omega}_m \\
&= (a_{Jn}a_J - a_{Jn} - \frac{1}{2}a_J + \frac{1}{2})\lambda_J + (a_{Jn} - a_{Jn}a_J - \frac{1}{2} + \frac{1}{2}a_J)i^*\lambda_J \\
&\quad +(a_{Jn} - y_K)\mu_K^J - (a_{Jn} - w_K)i^*\mu_K^J - \omega_K - i^*\tilde{\omega}_K \\
&= (a_{Jn}a_J - a_{Jn} - \frac{1}{2}a_J + \frac{1}{2})(\lambda_J - i^*\lambda_J) \\
&\quad +(a_{Jn} - y_K)\mu_K^J - (a_{Jn} - w_K)i^*\mu_K^J - \omega_K - i^*\tilde{\omega}_K \\
&= \frac{1}{2}(2a_{Jn} - 1)(a_J - 1)(\lambda_J - i^*\lambda_J) + (1 - y_K)(\mu_K^J + i^*\mu_K^J) \\
&\quad +(a_{Jn} - 1)(\mu_K^J - i^*\mu_K^J) + (y_K + w_K - 2)i^*\mu_K^J - \omega_K - i^*\tilde{\omega}_K.
\end{aligned}$$

By the representation of λ_J

$$\begin{aligned}
\mu_K^J - i^*\mu_K^J &= \frac{1}{2}\text{BW-}\lim_{J \ni j \to \infty}\{\lambda_J - i^*\lambda_J - (2a_{Jj} - 1)(\lambda_j - i^*\lambda_j)\} \\
&= (1 - a_J)(\lambda_J - i^*\lambda_J).
\end{aligned}$$

After all

$$\alpha_{nJ} - \underline{\mu}_K^n = \frac{1}{2}(a_J - 1)(\lambda_J - i^*\lambda_J) + (1 - y_K)(\mu_K^J + i^*\mu_K^J) - \omega_K - i^*\tilde{\omega}_K.$$

By Lemma 6.1 and the representation of $\lambda_n - i^*\lambda_n$

$$\begin{aligned}
\lambda_n - i^*\lambda_n &= 2(\alpha_{nJ} - i^*\alpha_{nJ}) + \beta_J^n - i^*\beta_J^n - \gamma_J^n + i^*\gamma_J^n \\
&= (2a_{Jn} - 1)(\lambda_J - i^*\lambda_J) + 2(\underline{\mu}_K^n - i^*\underline{\mu}_K^n).
\end{aligned}$$

We get

$$\begin{aligned}
&\alpha_{nJ} - \underline{\mu}_K^n - i^*(\alpha_{nJ} - \underline{\mu}_K^n) \\
&= (a_J - 1)(\lambda_J - i^*\lambda_J) - (\omega_K - i^*\omega_K) + (\tilde{\omega}_K - i^*\tilde{\omega}_K) \\
&= \frac{1}{2}(2a_{Jn} - 1)(\lambda_J - i^*\lambda_J) - (\beta_J^n - i^*\beta_J^n - \gamma_J^n + i^*\gamma_J^n).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&(a_J - a_{Jn} - \frac{1}{2})(\lambda_J - i^*\lambda_J) = (a_J - a_{Jn} - \frac{1}{2})(\beta_J^n + \gamma_J^n - i^*(\beta_J^n + \gamma_J^n)) \\
&= (\omega_K - i^*\omega_K) - (\tilde{\omega}_K - i^*\tilde{\omega}_K) - (\beta_J^n - i^*\beta_J^n - \gamma_J^n + i^*\gamma_J^n).
\end{aligned}$$

It follows that

$$\begin{aligned}
&(a_J - a_{Jn} + \frac{1}{2})\beta_J^n + (a_J - a_{Jn} - \frac{3}{2})\gamma_J^n - \omega_K + \tilde{\omega}_K \\
&= i^*((a_J - a_{Jn} + \frac{1}{2})\beta_J^n + (a_J - a_{Jn} - \frac{3}{2})\gamma_J^n - \omega_K + \tilde{\omega}_K).
\end{aligned}$$

Proposition 6.2. *When* $a_J(\varphi) = 1$,

(1) $\lim_{J\ni j\to\infty}(||\tilde{\beta}_{Jj}^n||^2 + <\beta_{Jj}^n, \tilde{\beta}_{Jj}^n>) = \lim_{J\ni j\to\infty}(||\tilde{\gamma}_{Jj}^n||^2 + <\gamma_{Jj}^n, \tilde{\gamma}_{Jj}^n>) = 0,$

(2) $\lim_{J\ni j\to\infty}(||\tilde{\beta}_{Jj}^n||^2 + <\tilde{\beta}_{Jj}^n, \tilde{\gamma}_{Jj}^n>) = \lim_{J\ni j\to\infty}(||\tilde{\gamma}_{Jj}^n||^2 + <\tilde{\beta}_{Jj}^n, \tilde{\gamma}_{Jj}^n>) = 0,$

(3) $\lim_{J\ni j\to\infty} <\beta_{Jj}^n, \beta_{Jj}^n + \gamma_{Jj}^n> = a_{Jn}\frac{||\varphi||^2}{2}.$

Proof. By Lemma 6.6 we have

$$\begin{aligned}
&\lim_{J\ni j\to\infty}(||\tilde{\beta}_{Jj}^n||^2 + ||\tilde{\gamma}_{Jj}^n||^2 + 2<\tilde{\beta}_{Jj}^n, \tilde{\gamma}_{Jj}^n>) \\
&= \lim_{J\ni j\to\infty}||\tilde{\beta}_{Jj}^n + \tilde{\gamma}_{Jj}^n||^2 = \lim_{J\ni j\to\infty}(a_J - 2a_{Jj} + 1)\frac{||\varphi||^2}{4} = 0, \\
&\therefore \lim_{J\ni j\to\infty}(||\tilde{\beta}_{Jj}^n||^2 - ||\tilde{\gamma}_{Jj}^n||^2) = 0.
\end{aligned}$$

By Lemma 6.7 we have

$$0 = \lim_{J\ni j\to\infty}(<\beta_{Jj}^n, \tilde{\beta}_{Jj}^n> - <\tilde{\beta}_{Jj}^n, \tilde{\gamma}_{Jj}^n + \tilde{\beta}_{Jj}^n - \tilde{\beta}_{Jj}^n>)$$

$$= \lim_{J\ni j\to\infty}(<\beta_{Jj}^n, \tilde{\beta}_{Jj}^n> + ||\tilde{\beta}_{Jj}^n||^2) = \lim_{J\ni j\to\infty}(<\gamma_{Jj}^n, \tilde{\gamma}_{Jj}^n> + ||\tilde{\gamma}_{Jj}^n||^2)$$

$$\lim_{J\ni j\to\infty}(||\tilde{\beta}_{Jj}^n||^2 + <\tilde{\beta}_{Jj}^n, \tilde{\gamma}_{Jj}^n>) = \lim_{J\ni j\to\infty} <\tilde{\beta}_{Jj}^n, \tilde{\beta}_{Jj}^n + \tilde{\gamma}_{Jj}^n> = 0$$

$$\lim_{J\ni j\to\infty}(||\tilde{\gamma}_{Jj}^n||^2 + <\tilde{\gamma}_{Jj}^n, \tilde{\beta}_{Jj}^n>) = \lim_{J\ni j\to\infty} <\tilde{\gamma}_{Jj}^n, \tilde{\beta}_{Jj}^n + \tilde{\gamma}_{Jj}^n> = 0$$

$$\lim_{J\ni j\to\infty} < \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} >= a_{Jn}\frac{||\varphi||^2}{2}.$$ □

Proposition 6.3. *When* $a_J(\varphi) = \frac{1}{2}$,

(1) $\beta^n_{Jm} + \gamma^n_{Jm} = \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm} = \frac{1}{2}\lambda_m$,

(2) $\lim_{J\ni j\to\infty} < \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} >= \lim_{J\ni j\to\infty} < \tilde{\beta}^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >= \frac{1}{16}||\varphi||^2$,

(3) $\lim_{J\ni j\to\infty}(||\beta^n_{Jj}||^2 - ||\gamma^n_{Jj}||^2) = \lim_{J\ni j\to\infty}(||\tilde{\beta}^n_{Jj}||^2 - ||\tilde{\gamma}^n_{Jj}||^2) = 0.$

Proof. (1) By Proposition 5.1 $\lambda_J = i\,{}^*\lambda_J$. Using

$$\lambda_J = \beta^n_J + \gamma^n_J = \beta^n_{Jm} + \gamma^n_{Jm} + i\,{}^*(\tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm}),\ \text{where } \beta^n_{Jm} + \gamma^n_{Jm}, \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm} \in \Lambda_{cm},$$

we have $\beta^n_{Jm} + \gamma^n_{Jm} = \tilde{\beta}^n_{Jm} + \tilde{\gamma}^n_{Jm} = \frac{1}{2}\lambda_m$ (Lemma 6.5).

(2) By Lemma 6.7

$$\lim_{J\ni j\to\infty} < \tilde{\beta}^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >= \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \tilde{\gamma}^n_{Jj} > + < \beta^n_{Jj}, \tilde{\beta}^n_{Jj} >) = \lim_{J\ni j\to\infty} < \beta^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} >$$

$$= \lim_{J\ni j\to\infty}(< \beta^n_{Jj}, \tilde{\gamma}^n_{Jj} > + < \gamma^n_{Jj}, \tilde{\gamma}^n_{Jj} >) = \lim_{J\ni j\to\infty} < \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj}, \tilde{\gamma}^n_{Jj} >$$

$$= \lim_{J\ni j\to\infty}(< \gamma^n_{Jj}, \tilde{\gamma}^n_{Jj} > + < \gamma^n_{Jj}, \tilde{\beta}^n_{Jj} >) = \lim_{J\ni j\to\infty} < \gamma^n_{Jj}, \beta^n_{Jj} + \gamma^n_{Jj} >$$

$$= \frac{1}{2}\lim_{J\ni j\to\infty} < \beta^n_{Jj} + \gamma^n_{Jj}, \tilde{\beta}^n_{Jj} + \tilde{\gamma}^n_{Jj} >= \frac{1}{16}||\varphi||^2\ \text{(Lemma 6.6)}.$$

(3) Above equalities show that

$$\lim_{J\ni j\to\infty}(||\beta^n_{Jj}||^2 - ||\gamma^n_{Jj}||^2) = \lim_{J\ni j\to\infty}(||\tilde{\beta}^n_{Jj}||^2 - ||\tilde{\gamma}^n_{Jj}||^2) = 0.$$ □

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