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A Sequence of Behavior Spaces and the Structure of Its Convergent Space II

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Abstract

It should be expected to construct behavior spaces with a certain period condition on a Riemann surface of infinite genus, because behavior spaces play a central role in a formulation of Riemann-Roch's and Abel's theorem on an open Riemann surface. For the construction a sequence of behavior spaces and a certain weakly convergent space will be used. Systematical investigation of the structure of its convergent space is carried. Our discourse treats of some subjects on admissible sequences, complex conjugate behavior spaces and linear mapping from a subspace of holomorphic differentials to a subspace of anti-holomorphic differentials.

Key Words: *Riemann surface; harmonic differentials; behavior spaces.*

1. Introduction

Let $\Lambda = \Lambda(R)$ be a real Hilbert space which consists of square integrable complex differentials on a Riemann surface R. Its inner product is given by as follows:

$$< \omega, \sigma >= \Re(\omega, \sigma),\ (\omega, \sigma) = \iint_R \omega \wedge {}^*\bar{\sigma},$$

where $\bar{\sigma}$ is the complex conjugate differential of σ, ${}^*\bar{\sigma}$ is the conjugate differential of $\bar{\sigma}$ and $\Re(\omega, \sigma)$ means the real part of the complex inner product (ω, σ). Let $\Lambda_h,\ \Lambda_a$, and $\Lambda_{\bar{a}}$ be the subspaces of Λ consisting of harmonic, holomorphic, anti-holomorphic differentials respectively. For a subspace Λ_x of Λ_h $\Lambda_x^{\perp}$ denotes the orthogonal complement of Λ_x in Λ_h and ${}^*\Lambda_x = \{{}^*\omega : \omega \in \Lambda_x\}$. We say Λ_x a general behavior space, if $\Lambda_x^{\perp} = i\,{}^*\Lambda_x = \{i\,{}^*\omega : \omega \in \Lambda_x\}$. For a sequence of general behavior spaces $\{\Lambda_n\}_{n\in\mathbf{N}}$, consider the following subspaces:

$\Lambda_s = \{\lambda \in \Lambda_h :^{\exists} \lambda_n \in \Lambda_n\ (n \in \mathbf{N}) \text{ s.t. } \operatorname*{s\text{-}lim}_{n\to\infty} \lambda_n = \lambda\}$,

$\tilde{\Lambda}_{bw} = \{\lambda \in \Lambda_h :^{\exists} J \subset \mathbf{N},\ {}^{\exists}\lambda_j \in \Lambda_j\ (j \in J) \text{ s.t. } \operatorname*{BW\text{-}lim}_{J\ni j\to\infty} \lambda_j = \lambda\}$,

$\tilde{\Lambda}'_{bw} = \{\sum_{i=1}^{\ell} c_i\,\lambda_i : \lambda_i \in \tilde{\Lambda}_{bw}, c_i \in \mathbf{R}\}$,

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$\Lambda_{bw} = Cl(\tilde{\Lambda}'_{bw})$,
where $\text{s-}\lim_{n\to\infty} \lambda_n$ means strong (normed) convergence and $\text{BW-}\lim_{J\ni j\to\infty} \lambda_j$ means weak and norm bounded convergence (cf. [3]).

We have shown in [3] the following.

Proposition 1.1.

$$\Lambda_h = \Lambda_{bw} \oplus i\,^*\Lambda_s = \Lambda_s \oplus i\,^*\Lambda_s \oplus (\Lambda_{bw} \cap i\,^*\Lambda_{bw})$$

If $\Lambda_{bw} \cap i\,^*\Lambda_{bw}$ can be divided into the spaces of holomorphic differentials and anti-holomorphic differentials evenly (cf. [3]), a behavior space can be constructed. We will observe the structure of $\Lambda_{bw} \cap i\,^*\Lambda_{bw}$.

2. Decomposition of sequences and the coefficients relations

Let us begin to introduce our circumstances and notations. Suppose $\{\Lambda_n\}_{n\in\mathbf{N}}$ is a sequence of general behavior spaces. By the orthogonal decomposition $\Lambda_h = \Lambda_n + i\,^*\Lambda_n$, for each $\varphi \in \Lambda_a$ we can find uniquely $\lambda_n(\varphi) \in \Lambda_n$ such that

$$\varphi = \lambda_n(\varphi) + i\,^*\lambda_n(\varphi).$$

Further for $\lambda_m(\varphi) \in \Lambda_m$ we can find a real number $a_{mn}(\varphi)$ and $\mu_n^m(\varphi) \in \Lambda_n$ such that

$$\lambda_m(\varphi) = a_{mn}(\varphi)\lambda_n(\varphi) + \{1 - a_{mn}(\varphi)\}i\,^*\lambda_n(\varphi) + \mu_n^m(\varphi) - i\,^*\mu_n^m(\varphi),$$

where $\mu_n^m(\varphi)$ is orthogonal to $\lambda_n(\varphi)$ (*cf.* [3]). Similarly we have

$$\overline{\varphi} = \lambda_n(\overline{\varphi}) - i\,^*\lambda_n(\overline{\varphi}), \text{where } \lambda_n(\overline{\varphi}) \in \Lambda_n,$$

$$\lambda_m(\overline{\varphi}) = a_{mn}(\overline{\varphi})\ \lambda_n(\overline{\varphi}) - \{1 - a_{mn}(\overline{\varphi})\}i\,^*\lambda_n(\overline{\varphi}) + \mu_n^m(\overline{\varphi}) + i\,^*\mu_n^m(\overline{\varphi}),$$

where $\mu_n^m(\overline{\varphi}) \in \Lambda_n$ and $\mu_n^m(\overline{\varphi})$ is orthogonal to $\lambda_n(\overline{\varphi})$. For an infinite subsequence J, if $\text{BW-}\lim_{J\ni j\to\infty} \lambda_j(\varphi) = \lambda_J(\varphi)$ exists, we say that J is admissible for φ, and if $\text{BW-}\lim_{J\ni j\to\infty} \lambda_j(\overline{\varphi}) = \lambda_J(\overline{\varphi})$ exists, we say that J is admissible for $\overline{\varphi}$. Then we remarked in [3] the following: there exist

$$\text{BW-}\lim_{J\ni j\to\infty} \mu_n^j(\varphi) = \mu_n^J(\varphi) \in \Lambda_n \text{ and } \text{BW-}\lim_{J\ni j\to\infty} \mu_n^j(\overline{\varphi}) = \mu_n^J(\overline{\varphi}) \in \Lambda_n$$

and they are orthogonal to $\lambda_n(\varphi)$ and $\lambda_n(\overline{\varphi})$ respectively. We have

$$\lambda_J(\varphi) = a_{Jn}(\varphi)\lambda_n(\varphi) + \{1 - a_{Jn}(\varphi)\}i\,^*\lambda_n(\varphi) + \mu_n^J(\varphi) - i\,^*\mu_n^J(\varphi),$$

$$\lambda_J(\overline{\varphi}) = a_{Jn}(\overline{\varphi})\lambda_n(\overline{\varphi}) - \{1 - a_{Jn}(\overline{\varphi})\}i\,^*\lambda_n(\overline{\varphi}) + \mu_n^J(\overline{\varphi}) + i\,^*\mu_n^J(\overline{\varphi}).$$

Furthermore, there exist the limits

$$\lim_{J \ni j \to \infty} a_{Jj}(\varphi) = a_J(\varphi), \lim_{J \ni j \to \infty} a_{Jj}(\overline{\varphi}) = a_J(\overline{\varphi}).$$

Lemma 2.1.

$$\|\lambda_J(\varphi)\|^2 = \frac{1}{2}a_J(\varphi)\|\varphi\|^2, \; < i\,^*\lambda_J(\varphi), \lambda_J(\varphi) >= \frac{1}{2}\{1 - a_J(\varphi)\}\|\varphi\|^2,$$

$$\|\mu_n^J(\varphi) - i\,^*\mu_n^J(\varphi)\|^2 = \frac{1}{2}\{a_J(\varphi) - 1 + 2a_{Jn}(\varphi) - 2a_{Jn}(\varphi)^2\}\|\varphi\|^2,$$

$$\|\lambda_J(\overline{\varphi})\|^2 = \frac{1}{2}a_J(\overline{\varphi})\|\overline{\varphi}\|^2, \; < i\,^*\lambda_J(\overline{\varphi}), \lambda_J(\overline{\varphi}) >= -\frac{1}{2}\{1 - a_J(\overline{\varphi})\}\|\overline{\varphi}\|^2,$$

$$\|\mu_n^J(\overline{\varphi}) + i\,^*\mu_n^J(\overline{\varphi})\|^2 = \frac{1}{2}\{a_J(\overline{\varphi}) - 1 + 2a_{Jn}(\overline{\varphi}) - 2a_{Jn}(\overline{\varphi})^2\}\|\overline{\varphi}\|^2.$$

Proof. For simplicity we denote $\lambda_J(\varphi)$ and $\lambda_n(\varphi)$ by λ_J and λ_n respectively. Then we get

$$\|\lambda_J\|^2 = \lim_{J \ni n \to \infty} < \lambda_J, \lambda_n >= \lim_{J \ni n \to \infty} a_{Jn}\|\lambda_n\|^2 = \frac{1}{2}a_J\|\varphi\|^2,$$

$$< i\,^*\lambda_J, \lambda_J >= \lim_{J \ni n \to \infty} < i\,^*\lambda_J, \lambda_n >= \lim_{J \ni n \to \infty}(1 - a_{Jn})\|\lambda_n\|^2 = \frac{1}{2}(1 - a_J)\|\varphi\|^2,$$

$$4\|\mu_n^J - i\,^*\mu_n^J\|^2 = \|\lambda_J - i\,^*\lambda_J\|^2 - (2a_{Jn} - 1)^2\|\varphi\|^2 = \{2a_J - 2 + 4a_{Jn} - 4(a_{Jn})^2\}\|\varphi\|^2.$$

Similarly we get the corresponding relations for $\overline{\varphi}$. □

Since a sequence with bounded norms contains a weakly convergent subsequence, we can choose a subsequence K of J such that

$$\text{BW-}\lim_{K \ni k \to \infty} \mu_k^J(\varphi) = \mu_K^J(\varphi), \; \text{BW-}\lim_{K \ni k \to \infty} \mu_k^J(\overline{\varphi}) = \mu_K^J(\overline{\varphi}).$$

Lemma 2.2.

$$\mu_K^J(\varphi) - i\,^*\mu_K^J(\varphi) = \{1 - a_J(\varphi)\}\{\lambda_J(\varphi) - i\,^*\lambda_J(\varphi)\},$$

$$\mu_K^J(\overline{\varphi}) + i\,^*\mu_K^J(\overline{\varphi}) = \{1 - a_J(\overline{\varphi})\}\{\lambda_J(\overline{\varphi}) + i\,^*\lambda_J(\overline{\varphi})\}.$$

Proof. Note that

$$\mu_K^J(\varphi) - i\,^*\mu_K^J(\varphi) = \text{BW-}\lim_{K \ni k \to \infty}\{\mu_k^J(\varphi) - i^*\mu_k^J(\varphi)\}$$

$$= \text{BW-}\lim_{K \ni k \to \infty}[\lambda_J(\varphi) - a_{Jk}(\varphi)\lambda_k(\varphi) - \{1 - a_{Jk}(\varphi)\}i\,^*\lambda_k(\varphi)]$$

$$= \{1 - a_J(\varphi)\}\{\lambda_J(\varphi) - i\,^*\lambda_J(\varphi)\},$$

$$\mu^J_K(\overline{\varphi}) + i^{\,*}\mu^J_K(\overline{\varphi}) = \text{BW-}\lim_{K\ni k\to\infty}\{\mu^J_k(\overline{\varphi}) + i^{\,*}\mu^J_k(\overline{\varphi})\}$$
$$= \text{BW-}\lim_{K\ni k\to\infty}[\lambda_J(\overline{\varphi}) - a_{Jk}(\overline{\varphi})\lambda_k(\overline{\varphi}) + \{1 - a_{Jk}(\overline{\varphi})\}i^{\,*}\lambda_k(\overline{\varphi})]$$
$$= \{1 - a_J(\overline{\varphi})\}\{\lambda_J(\overline{\varphi}) + i^{\,*}\lambda_J(\overline{\varphi})\}.$$

□

Proposition 2.1.

$$\{a_J(\varphi) - 1\}\lambda_J(\varphi) + \mu^J_K(\varphi) \in \Lambda_a,$$

$$\{a_J(\overline{\varphi}) - 1\}\lambda_J(\overline{\varphi}) + \mu^J_K(\overline{\varphi}) \in \Lambda_{\bar{a}}.$$

Proof. By Lemma 2.2,

$$\mu^J_K(\varphi) + i^{\,*}\mu^J_K(\varphi) - 2i^{\,*}\mu^J_K(\varphi) = \{1 - a_J(\varphi)\}\{\lambda_J(\varphi) + i^{\,*}\lambda_J(\varphi) - 2i^{\,*}\lambda_J(\varphi)\}$$

$$= \{1 - a_J(\varphi)\}\varphi - 2\{1 - a_J(\varphi)\}i^{\,*}\lambda_J(\varphi).$$

Hence

$$2i^{\,*}[\{1 - a_J(\varphi)\}\lambda_J(\varphi) - \mu^J_K(\varphi)] = \{1 - a_J(\varphi)\}\varphi - \{\mu^J_K(\varphi) + i^{\,*}\mu^J_K(\varphi)\} \in \Lambda_a.$$

Similarly

$$\mu^J_K(\overline{\varphi}) - i^{\,*}\mu^J_K(\overline{\varphi}) + 2i^{\,*}\mu^J_K(\overline{\varphi}) = \{1 - a_J(\overline{\varphi})\}\{\lambda_J(\overline{\varphi}) - i^{\,*}\lambda_J(\overline{\varphi}) + 2i^{\,*}\lambda_J(\overline{\varphi})\}$$

$$= \{1 - a_J(\overline{\varphi})\}\overline{\varphi} + 2\{1 - a_J(\overline{\varphi})\}i^{\,*}\lambda_J(\overline{\varphi}).$$

Hence

$$2i^{\,*}[-\{1 - a_J(\overline{\varphi})\}\lambda_J(\overline{\varphi}) + \mu^J_K(\overline{\varphi})] = \{1 - a_J(\overline{\varphi})\}\overline{\varphi} - \{\mu^J_K(\overline{\varphi}) - i^{\,*}\mu^J_K(\overline{\varphi})\} \in \Lambda_{\bar{a}}.$$

□

Lemma 2.3.

$$< \mu^J_K(\varphi), \lambda_J(\varphi) >= \frac{1}{4}\{1 - a_J(\varphi)\}\{2a_J(\varphi) - 1\}\|\varphi\|^2,\ < \mu^J_K(\varphi), \varphi >= 0,$$

$$< \mu^J_K(\overline{\varphi}), \lambda_J(\overline{\varphi}) >= \frac{1}{4}\{1 - a_J(\overline{\varphi})\}\{2a_J(\overline{\varphi}) - 1\}\|\overline{\varphi}\|^2,\ < \mu^J_K(\overline{\varphi}), \overline{\varphi} >= 0.$$

Proof. We have

$$< \mu^J_K, \lambda_J >= \lim_{K\ni k\to\infty} < \mu^J_k, \lambda_J >= \lim_{K\ni k\to\infty} ||\mu^J_k||^2 = \lim_{K\ni k\to\infty} \frac{1}{2}||\mu^J_k - i^{\,*}\mu^J_k||^2$$

$$= \lim_{K\ni k\to\infty} \frac{1}{4}\{a_J + 2a_{Jk} - 1 - 2(a_{Jk})^2\}\|\varphi\|^2 = \frac{1}{4}(1 - a_J)(2a_J - 1)\|\varphi\|^2.$$

Similarly we have

$$< \mu_K^J(\varphi), \varphi >=< \mu_K^J(\varphi), \lambda_J(\varphi) + i\,^*\lambda_J(\varphi) >$$
$$= \lim_{K\ni k\to\infty}(< \mu_k^J(\varphi), \lambda_J(\varphi) > + < \mu_k^J(\varphi), i\,^*\lambda_J(\varphi) >)$$
$$= \lim_{K\ni k\to\infty}(\|\mu_k^J(\varphi)\|^2 - \|\mu_k^J(\varphi)\|^2) = 0,$$
$$< \mu_K^J, \lambda_J >= \lim_{K\ni k\to\infty} < \mu_k^J, \lambda_J >= \lim_{K\ni k\to\infty} ||\mu_k^J||^2 = \lim_{K\ni k\to\infty} \frac{1}{2}||\mu_k^J + i\,^*\mu_k^J||^2$$
$$= \lim_{K\ni k\to\infty} \frac{1}{4}\{a_J + 2a_{Jk} - 1 - 2(a_{Jk})^2\}\|\overline{\varphi}\|^2 = \frac{1}{4}(1-a_J)(2a_J-1)\|\overline{\varphi}\|^2,$$
$$< \mu_K^J(\overline{\varphi}), \overline{\varphi} >=< \mu_K^J(\overline{\varphi}), \lambda_J(\overline{\varphi}) - i\,^*\lambda_J(\overline{\varphi}) >$$
$$= \lim_{K\ni k\to\infty}(< \mu_k^J(\overline{\varphi}), \lambda_J(\overline{\varphi}) > - < \mu_k^J(\overline{\varphi}), i\,^*\lambda_J(\overline{\varphi}) >)$$
$$= \lim_{K\ni k\to\infty}(\|\mu_k^J(\overline{\varphi})\|^2 - \|\mu_k^J(\overline{\varphi})\|^2) = 0.$$

$\square$

Consider the following orthogonal decomposition:

$$\begin{aligned}\mu_K^J(\varphi) = \quad & a_{Kk}^J(\varphi)\mu_k^J(\varphi) + b_{Kk}^J(\varphi)i\,^*\mu_k^J(\varphi) + c_{Kk}^J(\varphi)\lambda_k(\varphi) + d_{Kk}^J(\varphi)i\,^*\lambda_k(\varphi) \\ & +\mu_{Kk}^J(\varphi) + i\,^*\nu_{Kk}^J(\varphi), \\ \{\mu_K^J(\overline{\varphi}) = \quad & a_{Kk}^J(\overline{\varphi})\mu_k^J(\overline{\varphi}) + b_{Kk}^J(\overline{\varphi})i\,^*\mu_k^J(\overline{\varphi}) + c_{Kk}^J(\overline{\varphi})\lambda_k(\overline{\varphi}) + d_{Kk}^J(\overline{\varphi})i\,^*\lambda_k(\overline{\varphi}) \\ & +\mu_{Kk}^J(\overline{\varphi}) + i\,^*\nu_{Kk}^J(\overline{\varphi})\},\end{aligned}$$

where $\mu_{Kk}^J(\varphi)$ $\{\mu_{Kk}^J(\overline{\varphi})\}$ and $\nu_{Kk}^J(\varphi)$ $\{\nu_{Kk}^J(\overline{\varphi}) \in \Lambda_k\}$ are orthogonal to $\mu_k^J(\varphi)$ $\{\mu_k^J(\overline{\varphi})\}$ and $\lambda_k(\varphi)$ $\{\lambda_k(\overline{\varphi})\}$ respectively.

Abbreviating the symbols (φ) and $(\overline{\varphi})$ in the notations, we have

$$\mu_K^J - i\,^*\mu_K^J$$
$$= (a_{Kk}^J - b_{Kk}^J)(\mu_k^J - i\,^*\mu_k^J) + (c_{Kk}^J - d_{Kk}^J)(\lambda_k - i\,^*\lambda_k) + (\mu_{Kk}^J - \nu_{Kk}^J) - i\,^*(\mu_{Kk}^J - \nu_{Kk}^J)$$
$$= (1-a_J)(\lambda_J - i\,^*\lambda_J) = (1-a_J)(2a_{Jk}-1)(\lambda_k - i\,^*\lambda_k) + 2(1-a_J)(\mu_k^J - i\,^*\mu_k^J),$$
$$\{\mu_K^J + i\,^*\mu_K^J$$
$$= (a_{Kk}^J + b_{Kk}^J)(\mu_k^J + i\,^*\mu_k^J) + (c_{Kk}^J + d_{Kk}^J)(\lambda_k + i\,^*\lambda_k) + (\mu_{Kk}^J + \nu_{Kk}^J) + i\,^*(\mu_{Kk}^J + \nu_{Kk}^J)$$
$$= (1-a_J)(\lambda_J + i\,^*\lambda_J) = (1-a_J)(2a_{Jk}-1)(\lambda_k + i\,^*\lambda_k) + 2(1-a_J)(\mu_k^J + i\,^*\mu_k^J)\}.$$

It follows that

$$a_{Kk}^J - b_{Kk}^J = 2(1-a_J),\ c_{Kk}^J - d_{Kk}^J = (1-a_J)(2a_{Jk}-1), \mu_{Kk}^J = \nu_{Kk}^J,$$
$$\{a_{Kk}^J + b_{Kk}^J = 2(1-a_J),\ c_{Kk}^J + d_{Kk}^J = (1-a_J)(2a_{Jk}-1), \mu_{Kk}^J = -\nu_{Kk}^J\}.$$

By Lemma 2.1, we have

$$c_K^J = \lim_{K\ni k\to\infty} c_{Kk}^J = \lim_{K\ni k\to\infty} \frac{<\mu_K^J, \lambda_k>}{||\lambda_k||^2} = \lim_{K\ni k\to\infty} \frac{2<\mu_k^J, \lambda_K>}{||\varphi||^2}$$

$$= \lim_{K\ni k\to\infty} \frac{1}{2}\{a_J + 2a_{Jk} - 1 - 2(a_{Jk})^2\} = \frac{1}{2}(1-a_J)(2a_J - 1).$$

We may assume that $a_K^J = \lim_{K\ni k\to\infty} a_{Kk}^J$, $\mu_{KK}^J = \text{BW-}\lim_{K\ni k\to\infty} \mu_{Kk}^J$, replacing K by its subsequence if necessary. We have the following representation:

$$b_K^J(\varphi) = \lim_{K\ni k\to\infty} b_{Kk}^J = a_K^J(\varphi) - 2\{1 - a_J(\varphi)\},$$

$$b_K^J(\overline{\varphi}) = \lim_{K\ni k\to\infty} b_{Kk}^J = -a_K^J(\overline{\varphi}) + 2\{1 - a_J(\overline{\varphi})\},$$

$$d_K^J(\varphi) = \lim_{K\ni k\to\infty} d_{Kk}^J = -\frac{1}{2}\{1 - a_J(\varphi)\}\{2a_J(\varphi) - 1\},$$

$$d_K^J(\overline{\varphi}) = \lim_{K\ni k\to\infty} d_{Kk}^J = \frac{1}{2}\{1 - a_J(\overline{\varphi})\}\{2a_J(\overline{\varphi}) - 1\}.$$

It follows that

Proposition 2.2.

$$\|\mu_K^J(\varphi)\|^2 = \frac{1}{4}a_K^J(\varphi)\{1 - a_J(\varphi)\}\{2a_J(\varphi) - 1\}\|\varphi\|^2,$$

$$\|\mu_K^J(\overline{\varphi})\|^2 = \frac{1}{4}a_K^J(\overline{\varphi})\{1 - a_J(\overline{\varphi})\}\{2a_J(\overline{\varphi}) - 1\}\|\overline{\varphi}\|^2.$$

Proof. We have

$$\|\mu_K^J\|^2 = \lim_{K\ni k\to\infty} <\mu_K^J, \mu_k^J> = \lim_{K\ni k\to\infty} a_k^J \|\mu_k^J\|^2 = a_K^J(3a_J - 1 - 2a_J^2)\frac{1}{4}\|\varphi\|^2.$$

□

For $\psi_2 = \lambda_J(\varphi) - i\,{}^*\lambda_J(\varphi)$, we have

Proposition 2.3.

$$\lambda_n(\psi_2) = \{2a_{Jn}(\varphi) - 1\}\lambda_n(\varphi) + 2\mu_n^J(\varphi),\ \|\lambda_n(\psi_2)\|^2 = \frac{1}{2}\{2a_J(\varphi) - 1\}\|\varphi\|^2,$$

$$\lambda_K(\psi_2) = \{2a_J(\varphi) - 1\}\lambda_J(\varphi) + 2\mu_K^J(\varphi),$$

$$\|\lambda_K(\psi_2)\|^2 = \frac{1}{2}\{2a_J(\varphi) - 1\}[\{2a_J(\varphi) - 1\}\{2 - a_J(\varphi)\} + 2a_K^J(\varphi)\{1 - a_J(\varphi)\}]\|\varphi\|^2.$$

Proof. From

$$\lambda_J(\varphi) - i\,^*\lambda_J(\varphi) = \{2a_{Jn}(\varphi) - 1\}\{\lambda_n(\varphi) - i\,^*\lambda_n(\varphi)\} + 2\{\mu_n^J(\varphi) - i\,^*\mu_n^J(\varphi)\},$$

it is clear that

$$\lambda_n(\psi_2) = \{2a_{Jn}(\varphi) - 1\}\lambda_n(\varphi) + 2\mu_n^J(\varphi).$$

Hence

$$\begin{aligned}\|\lambda_n(\psi_2)\|^2 &= [\{2a_{Jn}(\varphi) - 1\}^2 + 2\{a_J(\varphi) + 2a_{Jn}(\varphi) - 1 - 2a_{Jn}(\varphi)^2\}]||\lambda_n(\varphi)||^2 \\ &= (2a_J(\varphi) - 1)||\lambda_n(\varphi)||^2, \\ \lambda_K(\psi_2) &= \text{BW-}\lim_{K\ni k\to\infty} \lambda_k(\psi_2) = \{2a_J(\varphi) - 1\}\lambda_J(\varphi) + 2\mu_K^J(\varphi).\end{aligned}$$

Next

$$\|\lambda_K(\psi_2)\|^2 = \lim_{K\ni k\to\infty} < \{2a_J(\varphi) - 1\}\lambda_J(\varphi) + 2\mu_K^J, \{2a_{Jk}(\varphi) - 1\}\lambda_k(\varphi) + 2\mu_k^J(\varphi) >$$

$$= \lim_{K\ni k\to\infty}[\{2a_J(\varphi){-}1)a_{Jk}(\varphi)\{2a_{Jk}(\varphi){-}1\}{+}2a_{Jk}^J(\varphi)\{a_J(\varphi){+}2a_{Jk}(\varphi){-}1{-}2a_{Jk}(\varphi)^2\}$$

$$+\{2a_{Jk}(\varphi){-}1\}^2\{1{-}a_J(\varphi)\}{+}\{2a_J(\varphi){-}1\}\{a_J(\varphi){+}2a_{Jk}(\varphi){-}1{-}2a_{Jk}(\varphi)^2)\}]||\lambda_k(\varphi)||^2$$

$$= \{2a_J(\varphi){-}1\}[\{2a_J(\varphi){-}1\}{-}2a_K^J(\varphi)\{a_J(\varphi){-}1\}{+}\{2a_J(\varphi){-}1\}\{1{-}a_J(\varphi)\}]\frac{1}{2}||\varphi||^2$$

$$= \{2a_J(\varphi) - 1\}[\{2a_J(\varphi) - 1\}\{2 - a_J(\varphi)\} + 2a_K^J(\varphi)\{1 - a_J(\varphi)\}]\frac{1}{2}||\varphi||^2.$$

□

Corollary 2.1.

If $a_J(\varphi) = \dfrac{1}{2}$, *it holds* $\psi_2 = 0$.

If $a_J(\varphi) = 1$, *it holds* $\lambda_K(\psi_2) = \lambda_J(\varphi)$, $a_K(\psi_2) = 1$, *and* $\mu_K^J(\psi_2) = 0$.

Proof. For $a_J(\varphi) = \frac{1}{2}$ or $a_J(\varphi) = 1$, by Proposition 2.2,

$$||\mu_K^J(\varphi)||^2 = \frac{1}{4}a_K^J(\varphi)\{2a_J(\varphi) - 1\}\{1 - a_J(\varphi)\}||\varphi||^2 = 0.$$

Hence $\mu_K^J(\varphi) = 0$.

If $a_J(\varphi) = \frac{1}{2}$, by Proposition 2.3, $\lambda_n(\psi_2) = 0$ and $\psi_2 = 0$. If $a_J(\varphi) = 1$, by Lemma 2.1 and Proposition 2.3,

$$a_K(\psi_2) = \frac{2\|\lambda_K(\psi_2)\|}{\|\psi_2\|} = \{2a_J(\varphi) - 1\}\{2 - a_J(\varphi)\} + 2a_K^J(\varphi)\{1 - a_J(\varphi)\} = 1.$$

□

3. Admissible sequences

For the guarantee of norm bounded weak convergence of every concerned differential sequences we choose a certain subsequence of general behavior spaces $\{\Lambda_n\}_{n\in\mathbf{N}}$. Let us take an orthonormal basis $\{\varphi_k\}$ of the space Λ_a. Then for $k \neq \ell$,

$$0 =< \varphi_k, \varphi_\ell >=< \lambda_n(\varphi_k)+i\,{}^*\lambda_n(\varphi_k), \lambda_n(\varphi_\ell)+i\,{}^*\lambda_n(\varphi_\ell) >= 2 < \lambda_n(\varphi_k), \lambda_n(\varphi_\ell) >$$

and similarly $0 =< \lambda_n(\overline{\varphi}_k), \lambda_n(\overline{\varphi}_\ell) >$. At first, take a subsequence $\underline{J}_1$ of $\mathbf{N}$, which is admissible for φ_1, and take a subsequence $J_1 = (n_{11}, n_{12}, n_{13}...)$ of $\underline{J}_1$, which is admissible for $\overline{\varphi}_1$. Next take a subsequence $\underline{J}_2$ of J_1, which is admissible for φ_2, and take a subsequence $J_2 = (n_{21}, n_{22}, n_{23}...)$ of $\underline{J}_2$, which is admissible for $\overline{\varphi}_2$. Repeating these process we get subsequences $J_m = (n_{m1}, n_{m2}, n_{m3}...)$. Let $J = (n_{11}, n_{22}, n_{33}...)$. Then for any $\varphi = \sum_{i=1} a_i\varphi_i \in \Lambda_a$ which converges by norm sense,

$$\text{BW-}\lim_{J\ni j\to\infty} \lambda_j(\varphi) = \text{BW-}\lim_{J\ni j\to\infty} \sum_{i=1} a_i\lambda_j(\varphi_i) = \sum_{i=1} a_i\lambda_J(\varphi_i),$$

where $\varphi = \lambda_j(\varphi) + i\,{}^*\lambda_j(\varphi), \lambda_j(\varphi) \in \Lambda_j,$ and $\|\lambda_J(\varphi_i)\| \le 1$.

Hence J is admissible for both φ and $\overline{\varphi}$. J is also admissible for both φ and $\overline{\varphi}$ with respect to the complex conjugate behavior spaces $\{\overline{\Lambda}_n(= \{\overline{\omega} : \omega \in \Lambda_n\})\}_{n\in\mathbf{N}}$. For any harmonic differential ω we put

$$\lambda_n(\omega) = \{\lambda_n(\omega + i\,{}^*\omega) + \lambda_n(\omega - i\,{}^*\omega)\}/2.$$

Then there exists $\text{BW-}\lim_{J\ni j\to\infty} \lambda_j(\omega) = \lambda_J(\omega)$.

Let Λ_s and Λ_{bw} denote the notations for the sequence of behavior spaces $\{\Lambda_j\}_{j\in J}$ instead of $\{\Lambda_n\}_{n\in\mathbf{N}}$. Then the complex conjugate spaces $\overline{\Lambda}_s$ and $\overline{\Lambda}_{bw}$ coincide with those for the sequence of behavior spaces $\{\overline{\Lambda}_j\}_{j\in J}$. Now let $\Omega_a = \Lambda_{bw} \cap i\,{}^*\Lambda_{bw} \cap \Lambda_a$, $\Omega_{\bar{a}} = \Lambda_{bw} \cap i\,{}^*\Lambda_{bw} \cap \Lambda_{\bar{a}}$. Then $\Omega_{a.5} = \{\varphi \in \Lambda_a; a_J(\varphi) = 0.5\}$ is a subspace of Ω_a and $\Omega_{\bar{a}.5} = \{\varphi \in \Lambda_{\bar{a}}; a_J(\varphi) = 0.5\}$ is a subspace of $\Omega_{\bar{a}}$. Set $\Omega_{a0} = \Omega_a \cap \Omega_{a.5}^{\perp}$, $\Omega_{\bar{a}0} = \Omega_{\bar{a}} \cap \Omega_{\bar{a}.5}^{\perp}$.

Proposition 3.1. *When* $\varphi_0 \in \Omega_{a0}$ *and* $\varphi_0 \neq 0$ ($\overline{\varphi_0} \in \Omega_{\bar{a}0}$ *and* $\overline{\varphi_0} \neq 0$), *set* $\theta(\varphi_0) = \lambda_J(\varphi_0) - \frac{1}{2}\varphi_0$, $(\theta(\overline{\varphi_0}) = \lambda_J(\overline{\varphi_0}) - \frac{1}{2}\overline{\varphi_0})$. *Then*

(1) $\theta(\varphi_0) \in \Omega_{\bar{a}0}$, $\theta(\overline{\varphi_0}) \in \Omega_{a0}$,

(2) $\lambda_J(\theta(\varphi_0)) = \{a_J(\varphi_0) - \dfrac{1}{2}\}\lambda_J(\varphi_0) + \mu_J^J(\varphi_0)$,

$\quad\lambda_J(\theta(\overline{\varphi_0})) = \{a_J(\overline{\varphi_0}) - \dfrac{1}{2}\}\lambda_J(\overline{\varphi_0}) + \mu_J^J(\overline{\varphi_0})$,

(3) $a_J(\theta(\varphi_0)) = 5a_J(\varphi_0) - 2a_J(\varphi_0)^2 - 2 + 2a_J^J(\varphi_0)\{1 - a_J(\varphi_0)\}$,

$$a_J(\theta(\overline{\varphi_0})) = 5a_J(\overline{\varphi_0}) - 2a_J(\overline{\varphi_0})^2 - 2 + 2a_J^J(\overline{\varphi_0})\{1 - a_J(\overline{\varphi_0})\},$$

(4) $8 \lim_{J \ni j \to \infty} ||\mu_{Jj}^J(\varphi_0)||^2$

$$= \{3 - 2a_J^J(\varphi_0) - 2a_J(\varphi_0)\}\{a_J^J(\varphi_0) + a_J(\varphi_0) - 1\}\{1 - a_J(\varphi_0)\}\{2a_J(\varphi_0) - 1\}\|\varphi_0\|^2,$$

$8 \lim_{J \ni j \to \infty} ||\mu_{Jj}^J(\overline{\varphi_0})||^2$

$$= \{3 - 2a_J^J(\overline{\varphi_0}) - 2a_J(\overline{\varphi_0})\}\{a_J^J(\overline{\varphi_0}) + a_J(\overline{\varphi_0}) - 1\}\{1 - a_J(\overline{\varphi_0})\}\{2a_J(\overline{\varphi_0}) - 1\}\|\overline{\varphi_0}\|^2.$$

Proof. (1) Since $\lambda_J(\varphi_0) \in \Lambda_{bw}$ and $i^*\lambda_J(\varphi_0) = \varphi_0 - \lambda_J(\varphi_0) \in \Lambda_{bw}$, we have $\lambda_J(\varphi_0) \in \Lambda_{bw} \cap i^*\Lambda_{bw}$. Take an orthogonal decomposition:

$$\lambda_J(\varphi_0) = \sum_{i=1}^{4} \theta_i, \text{where } \theta_1 \in \Omega_{a0},\ \theta_2 \in \Omega_{a.5},\ \theta_3 \in \Omega_{\bar{a}0}, \text{and } \theta_4 \in \Omega_{\bar{a}.5}.$$

Then

$$\varphi_0 = \lambda_J(\varphi_0) + i^*\lambda_J(\varphi_0) = \sum_{i=1}^{4}(\theta_i + i^*\theta_i) = \sum_{i=1}^{2} 2\theta_i \text{ and } \theta_1 = \frac{1}{2}\varphi_0,\ \theta_2 = 0.$$

For any $\bar{\psi} \in \Omega_{\bar{a}.5}$ we have

$$< \lambda_J(\varphi_0), \bar{\psi} >= \text{BW-}\lim_{J \ni j \to \infty} < \lambda_j(\varphi_0), \bar{\psi} >= \text{BW-}\lim_{J \ni j \to \infty} < \lambda_j(\varphi_0), \lambda_j(\bar{\psi}) - i^*\lambda_j(\bar{\psi}) >$$

$$= \text{BW-}\lim_{J \ni j \to \infty} < \lambda_j(\varphi_0), \lambda_j(\bar{\psi}) >= \text{BW-}\lim_{J \ni j \to \infty} < \lambda_j(\varphi_0) + i^*\lambda_j(\varphi_0), \lambda_j(\bar{\psi}) >$$

$$= \text{BW-}\lim_{J \ni j \to \infty} < \varphi_0, \lambda_j(\bar{\psi}) >=< \varphi_0, \lambda_J(\bar{\psi}) >=< \varphi_0, \frac{1}{2}\bar{\psi} >= 0.$$

It follows $\theta_4 = 0$ and $\lambda_J(\varphi_0) = \frac{1}{2}\varphi_0 + \theta_3$. Hence $\theta(\varphi_0) = \theta_3 \in \Omega_{\bar{a}0}$.
(2) By Lemma 2.1, we have

$$||\theta_3||^2 = ||\lambda_J(\varphi_0)||^2 - \frac{1}{4}||\varphi_0||^2 = \frac{1}{4}\{2a_J(\varphi_0) - 1\}||\varphi_0||^2.$$

From

$$\theta_3 = \lambda_J(\varphi_0) - \frac{1}{2}\varphi_0 = \{a_{Jj}(\varphi_0) - \frac{1}{2}\}\{\lambda_j(\varphi_0) - i^*\lambda_j(\varphi_0)\} + \mu_j^J(\varphi_0) - i^*\mu_j^J(\varphi_0),$$

we get

$$\lambda_j(\theta_3) = \{a_{Jj}(\varphi_0) - \frac{1}{2}\}\lambda_j(\varphi_0) + \mu_j^J(\varphi_0) \text{ and } \lambda_J(\theta_3) = \{a_J(\varphi_0) - \frac{1}{2}\}\lambda_J(\varphi_0) + \mu_J^J(\varphi_0).$$

(3) It follows that

$$\lambda_J(\theta_3) = \{a_J(\varphi_0) - \frac{1}{2}\}[a_{Jj}(\varphi_0)\lambda_j(\varphi_0) + \{1 - a_{Jj}(\varphi_0)\}i^*\lambda_j(\varphi_0) + \mu_j^J(\varphi_0) - i^*\mu_j^J(\varphi_0)]$$

$$+a^J_{Jj}(\varphi_0)\mu^J_j(\varphi_0)+[a^J_{Jj}(\varphi_0)-2\{1-a_J(\varphi_0)\}]i^*\mu^J_j(\varphi_0)+c^J_{Jj}(\varphi_0)\lambda_j(\varphi_0)$$
$$+[c^J_{Jj}(\varphi_0)-\{1-a_J(\varphi_0)\}\{2a_{Jj}(\varphi_0)-1\}]i^*\lambda_j(\varphi_0)+\mu^J_{Jj}(\varphi_0)+i^*\mu^J_{Jj}(\varphi_0)$$
$$=[a_{Jj}(\varphi_0)\{a_J(\varphi_0)-\frac{1}{2}\}+c^J_{Jj}(\varphi_0)]\lambda_j(\varphi_0)+\{a_J(\varphi_0)-\frac{1}{2}+a^J_{Jj}(\varphi_0)\}\mu^J_j(\varphi_0)+\mu^J_{Jj}(\varphi_0)$$
$$+\{c^J_{Jj}(\varphi_0)+\frac{1}{2}-\frac{3}{2}a_{Jj}(\varphi_0)+a_J(\varphi_0)a_{Jj}(\varphi_0)\}i^*\lambda_j(\varphi_0)$$
$$+\{a^J_{Jj}(\varphi_0)-\frac{3}{2}+a_J(\varphi_0)\}i^*\mu^J_j(\varphi_0)+i^*\mu^J_{Jj}(\varphi_0).$$

On the other hand,

$$\lambda_J(\theta_3)=a_{Jj}(\theta_3)\lambda_j(\theta_3)-\{1-a_{Jj}(\theta_3)\}i^*\lambda_j(\theta_3)+\mu^J_j(\theta_3)+i^*\mu^J_j(\theta_3)$$
$$=a_{Jj}(\theta_3)\{a_{Jj}(\varphi_0)-\frac{1}{2}\}\lambda_j(\varphi_0)+a_{Jj}(\theta_3)\mu^J_j(\varphi_0)+\mu^J_j(\theta_3)$$
$$-\{1-a_{Jj}(\theta_3)\}\{a_{Jj}(\varphi_0)-\frac{1}{2}\}i^*\lambda_j(\varphi_0)-\{1-a_{Jj}(\theta_3)\}i^*\mu^J_j(\varphi_0)+i^*\mu^J_j(\theta_3).$$

From both of these,

$$\mu^J_j(\theta_3)=[a_{Jj}(\varphi_0)\{a_J(\varphi_0)-\frac{1}{2}\}+c^J_{Jj}(\varphi_0)-a_{Jj}(\theta_3)\{a_{Jj}(\varphi_0)-\frac{1}{2}\}]\lambda_j(\varphi_0)$$
$$+\{a_J(\varphi_0)-\frac{1}{2}+a^J_{Jj}(\varphi_0)-a_{Jj}(\theta_3)\}\mu^J_j(\varphi_0)+\mu^J_{Jj}(\varphi_0).$$

By Lemma 2.1,

$$\|\mu^J_j(\theta_3)\|^2=\frac{1}{4}\{a_J(\theta_3)-1+2a_{Jj}(\theta_3)-2a_{Jj}(\theta_3)^2\}\|\theta_3\|^2.$$

Note that

$$||\lambda_J(\theta_3)||^2=\frac{1}{2}a_J(\theta_3)||\theta_3||^2=\frac{1}{8}a_J(\theta_3)\{2a_J(\varphi_0)-1\}||\varphi_0||^2.$$

On the other hand, by Lemma 2.3,

$$||\lambda_J(\theta_3)||^2=\{a_J(\varphi_0)-\frac{1}{2}\}^2||\lambda_J(\varphi_0)||^2+\frac{1}{4}\{1-a_J(\varphi_0)\}\{2a_J(\varphi_0)-1\}^2||\varphi_0||^2+||\mu^J_J(\varphi_0)||^2$$
$$=[\frac{1}{2}a_J(\varphi_0)\{a_J(\varphi_0)-\frac{1}{2}\}^2+\frac{1}{4}\{1-a_J(\varphi_0)\}\{2a_J(\varphi_0)-1\}^2$$
$$+\frac{1}{4}a^J_J(\varphi_0)\{1-a_J(\varphi_0)\}\{2a_J(\varphi_0)-1\}]||\varphi_0||^2$$
$$=\frac{1}{4}\{2a_J(\varphi_0)-1\}\{a_J(\varphi_0)^2-\frac{1}{2}a_J(\varphi_0)+3a_J(\varphi_0)-2a_J(\varphi_0)^2-1+a^J_J(\varphi_0)-a^J_J(\varphi_0)a_J(\varphi_0)\}||\varphi_0||^2.$$

Since $a_J(\varphi_0) \neq \frac{1}{2}$, we have

$$a_J(\theta_3) = 5a_J(\varphi_0) - 2a_J(\varphi_0)^2 - 2 + 2a_J^J(\varphi_0)\{1 - a_J(\varphi_0)\},$$

$$1 - a_J(\theta_3) = \{3 - 2a_J(\varphi_0) - 2a_J^J(\varphi_0)\}\{1 - a_J(\varphi_0)\}.$$

(4) We have

$$\mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0) = 2[a_{Jj}^J(\varphi_0) - \{1 - a_J(\varphi_0)\}]\{\mu_j^J(\varphi_0) + i\,{}^*\mu_j^J(\varphi_0)\}$$
$$+[2c_{Jj}^J(\varphi_0) - \{1 - a_J(\varphi_0)\}\{2a_{Jj}(\varphi_0) - 1\}]\{\lambda_j(\varphi_0) + i\,{}^*\lambda_j(\varphi_0)\}$$
$$+2\{\mu_{Jj}^J(\varphi_0) + i\,{}^*\mu_{Jj}^J(\varphi_0)\}.$$

Hence

$$||\mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0)||^2 = 4[a_{Jj}^J(\varphi_0) - \{1 - a_J(\varphi_0)\}]^2||\mu_j^J(\varphi_0) + i\,{}^*\mu_j^J(\varphi_0))||^2$$
$$+[2c_{Jj}^J(\varphi_0) - \{1 - a_J(\varphi_0)\}\{2a_{Jj}(\varphi_0) - 1\}]^2||\lambda_j(\varphi_0) + i\,{}^*\lambda_j(\varphi_0)))||^2$$
$$+4||\mu_{Jj}^J(\varphi_0) + i\,{}^*\mu_{Jj}^J(\varphi_0)||^2$$
$$= 2[a_{Jj}^J(\varphi_0) - \{1 - a_J(\varphi_0)\}^2\{a_J(\varphi_0) - 1 + 2a_{Jj}(\varphi_0) - 2a_{Jj}(\varphi_0)^2\}]\|\varphi_0\|^2$$
$$+[2c_{Jj}^J(\varphi_0) - \{1 - a_J(\varphi_0)\}\{2a_{Jj}(\varphi_0) - 1\}^2]||\varphi_0||^2 + 8||\mu_{Jj}^J(\varphi_0)||^2$$
$$= 2\{a_J^J(\varphi_0) - 1 + a_J(\varphi_0)\}^2\{1 - a_J(\varphi_0)\}\{2a_J(\varphi_0) - 1\}\|\varphi_0\|^2 + 8\lim_{J \ni j \to \infty}||\mu_{Jj}^J(\varphi_0)||^2.$$

On the other hand,

$$||\mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0)||^2 = \lim_{J \ni j \to \infty} < \mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0), \mu_j^J(\varphi_0) + i\,{}^*\mu_j^J(\varphi_0) >$$
$$= 2\lim_{J \ni j \to \infty}[a_{Jj}^J(\varphi_0) - \{1 - a_J(\varphi_0)\}||\mu_j^J(\varphi_0) + i\,{}^*\mu_j^J(\varphi_0)||^2]$$
$$= \{a_J^J(\varphi_0) + a_J(\varphi_0) - 1\}\{1 - a_J(\varphi_0)\}\{2a_J(\varphi_0) - 1\}\|\varphi_0\|^2.$$

Hence we get

$$8\lim_{J \ni j \to \infty}||\mu_{Jj}^J(\varphi_0)||^2$$
$$= \{3 - 2a_J^J(\varphi_0) - 2a_J(\varphi_0)\}\{a_J^J(\varphi_0) + a_J(\varphi_0) - 1\}\{1 - a_J(\varphi_0)\}\{2a_J(\varphi_0) - 1\}\|\varphi_0\|^2.$$

Similarly we can get the statement for $\overline{\varphi_0} \in \Omega_{\bar{a}0}$.

□

Proposition 3.2. *If a non-vanishing $\varphi_0 \in \Omega_{a0}$ satisfies $a_J(\varphi_0) + a_J^J(\varphi_0) \neq 1$, then φ_0 and $\varphi_1 = \theta(\theta(\varphi_0)) \in \Omega_{a0}$ are linearly independent, where θ is given in Proposition 3.1.*

Proof. If $\mu_J^J(\varphi_0) = i\,{}^*\mu_J^J(\varphi_0)$, by Lemma 2.2

$$0 = \mu_J^J(\varphi_0) - i\,{}^*\mu_J^J(\varphi_0) = 2\{1 - a_J(\varphi_0)\}\{\lambda_J(\varphi_0) - i\,{}^*\lambda_J(\varphi_0)\}$$

and $\lambda_J(\varphi_0) = i\,{}^*\lambda_J(\varphi_0) = \frac{1}{2}\varphi_0 \in \Omega_{a.5}$. This is a contradiction. Hence $\theta(\varphi_0) = \frac{1}{2}\{\lambda_J(\varphi_0) - i\,{}^*\lambda_J(\varphi_0)\} \in \Omega_{\bar{a}0}$ does not vanish. Similarly, for this $\theta(\varphi_0) \in \Omega_{\bar{a}0}$, we have $\varphi_1 = \theta\{\theta(\varphi_0)\} = \frac{1}{2}[\lambda_J\{\theta(\varphi_0)\} + i\,{}^*\lambda_J\{\theta(\varphi_0)\}] \neq 0$. By the assumption $a_J(\varphi_0) + a_J^J(\varphi_0) \neq 1$ and

$$||\mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0)||^2 = \{a_J^J(\varphi_0) + a_J(\varphi_0) - 1\}\{1 - a_J(\varphi_0)\}\{2a_J(\varphi_0) - 1\}\|\varphi_0\|^2,$$

$\mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0)$ does not vanish. By Lemma 2.3, we have

$$0 =< \mu_J^J(\varphi_0), \varphi_0 >=< i\,{}^*\mu_J^J(\varphi_0), i\,{}^*\varphi_0 >=< \mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0), \varphi_0 >$$

and $\mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0)$ is orthogonal to φ_0. Note that

$$\varphi_1 = \theta\{\theta(\varphi_0)\} = \lambda_J\{\theta(\varphi_0)\} - \frac{1}{2}\theta(\varphi_0) = \frac{1}{2}[\lambda_J\{\theta(\varphi_0)\} + i\,{}^*\lambda_J\{\theta(\varphi_0)\}]$$

$$= \frac{1}{2}[\{a_J(\varphi_0) - \frac{1}{2}\}\{\lambda_J(\varphi_0) + i\,{}^*\lambda_J(\varphi_0)\} + \mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0)]$$

$$= \frac{1}{2}[\{a_J(\varphi_0) - \frac{1}{2}\}\varphi_0 + \mu_J^J(\varphi_0) + i\,{}^*\mu_J^J(\varphi_0)].$$

It follows that φ_0 and φ_1 are linearly independent.

□

4. Complex conjugate behavior spaces

We are concerned to a correspondence of holomorphic and anti-holomorphic differentials in $\Lambda_{bw} \cap i\,{}^*\Lambda_{bw}$ (cf. [3]). We need a view point of complex conjugate. In previous sections we use the orthogonal decomposition with respect to $\{\Lambda_n\}$ and notations $\lambda_n,\ a_{mn}, \mu_n$ and so on. In this section we treat the orthogonal decomposition with respect to complex conjugate behavior spaces $\{\overline{\Lambda}_n\}$ and so use notations $\lambda'_n,\ a'_{mn}, \mu'_n$ with the symbol " $'$ ". For instance,

$$\varphi = \lambda'_n(\varphi) + i\,{}^*\lambda'_n(\varphi), \lambda'_n(\varphi) \in \overline{\Lambda_n},$$

$$\lambda'_m(\varphi) = a'_{mn}(\varphi)\lambda'_n(\varphi) + \{1 - a'_{mn}(\varphi)\}i\,{}^*\lambda'_n(\varphi) + (\mu')_n^m(\varphi) - i\,{}^*(\mu')_n^m(\varphi).$$

Let J be admissible for every holomorphic and anti-holomorphic differentials as in Section 3. We have

$$\lambda'_J(\varphi) = a'_{Jj}(\varphi)\lambda'_j(\varphi) + \{1 - a'_{Jj}(\varphi)\}i\,{}^*\lambda'_j(\varphi) + (\mu')_j^J(\varphi) - i\,{}^*(\mu')_j^J(\varphi),$$

where $(\mu')_j^J(\varphi) \in \bar{\Lambda}_j$ and $(\mu')_j^J(\varphi)$ is orthogonal to $\lambda'_j(\varphi)$.

Lemma 4.1.

$$\overline{\lambda'_j(\varphi)} = \lambda_j(\overline{\varphi}),\ a'_{mj}(\varphi) = a_{mj}(\overline{\varphi}),\ \overline{(\mu')^m_j(\varphi)} = \mu^m_j(\overline{\varphi}).$$

Proof. From $\overline{\varphi} = \overline{\lambda'_j(\varphi) + i\,^*\lambda'_j(\varphi)}$, $\overline{\lambda'_j(\varphi)} \in \Lambda_j$ and $\overline{\lambda'_j(\varphi)} = \lambda_j(\overline{\varphi})$, we have

$$\overline{\lambda_m(\overline{\varphi})} = a'_{mj}(\varphi)\overline{\lambda_j(\overline{\varphi})} + \{1 - a'_{mj}(\varphi)\}i\,^*\overline{\lambda_j(\overline{\varphi})} + (\mu')^m_j(\varphi) - i\,^*(\mu')^m_j(\varphi),$$

$$\lambda_m(\overline{\varphi}) = a'_{mj}(\varphi)\lambda_j(\overline{\varphi}) - \{1 - a'_{mj}(\varphi)\}i\,^*\lambda_j(\overline{\varphi}) + \overline{(\mu')^m_j(\varphi)} + i\,^*\overline{(\mu')^m_j(\varphi)}.$$

It follows that

$$a'_{mj}(\varphi) = a_{mj}(\overline{\varphi}),\ \overline{(\mu')^m_j(\varphi)} = \mu^m_j(\overline{\varphi}).$$

□

For $\varphi = \lambda'_j(\varphi) + i\,^*\lambda'_j(\varphi) = \lambda'_J(\varphi) + i\,^*\lambda'_J(\varphi)$, we set

$$\lambda'_{JR}(\varphi) = \frac{1}{2}\{\lambda'_J(\varphi) + \overline{\lambda'_J(\varphi)}\} = \frac{1}{2}\{\lambda_J(\overline{\varphi}) + \overline{\lambda_J(\overline{\varphi})}\} = \lambda_{JR}(\overline{\varphi}),$$

$$\lambda'_{JI}(\varphi) = \frac{1}{2}\{\lambda'_J(\varphi) - \overline{\lambda'_J(\varphi)}\} = \frac{1}{2}\{-\lambda_J(\overline{\varphi}) + \overline{\lambda_J(\overline{\varphi})}\} = -\lambda_{JI}(\overline{\varphi}),$$

$$\lambda'_{JI}(\varphi) = \omega_j + i\,^*\sigma_j, \text{where } \omega_j, \sigma_j \in \Lambda_j.$$

Since $\lambda'_{JI}(\varphi)$ is a pure imaginary differential, we have

$$0 =< \lambda'_{JI}(\varphi), i\,^*\lambda'_{JI}(\varphi) >=< \omega_j + i\,^*\sigma_j, i\,^*\omega_j + \sigma_j >= 2 < \omega_j, \sigma_j >,$$

$$\lambda'_{JI}(\varphi) + i\,^*\lambda'_{JI}(\varphi) = \omega_j + \sigma_j + i\,^*(\omega_j + \sigma_j).$$

Setting $\psi_3 = \lambda'_{JI}(\varphi) + i\,^*\lambda'_{JI}(\varphi)$, we have

Lemma 4.2.

$$\varphi = \lambda_J(\overline{\varphi}) + i\,^*\lambda_J(\overline{\varphi}) + 2\psi_3,\ \lambda'_J(\varphi) = \lambda_J(\overline{\varphi}) + 2(\omega_j + i\,^*\sigma_j).$$

Proof. Note that

$$\begin{aligned}
\varphi &= \lambda'_J(\varphi) + i\,^*\lambda'_J(\varphi) = \{\lambda'_{JR}(\varphi) + \lambda'_{JI}(\varphi)\} + i\,^*\{\lambda'_{JR}(\varphi) + \lambda'_{JI}(\varphi)\} \\
&= \{\lambda'_{JR}(\varphi) - \lambda'_{JI}(\varphi)\} + i\,^*\{\lambda'_{JR}(\varphi) - \lambda'_{JI}(\varphi)\} + 2\{\lambda'_{JI}(\varphi) + i\,^*\lambda'_{JI}(\varphi)\} \\
&= \lambda_J(\overline{\varphi}) + i\,^*\lambda_J(\overline{\varphi}) + 2\psi_3,
\end{aligned}$$

$$\lambda'_J(\varphi) = \lambda'_{JR}(\varphi) - \lambda'_{JI}(\varphi) + 2\lambda'_{JI}(\varphi) = \lambda_J(\overline{\varphi}) + 2(\omega_j + i\,^*\sigma_j).$$

□

Let

$$\text{BW-}\lim_{J \ni j \to \infty} \omega_j = \omega_J,\ \text{BW-}\lim_{J \ni j \to \infty} \sigma_j = \sigma_J.$$

Then

$$\lambda_j(\psi_3) = \omega_j + \sigma_j,\ \lambda_J(\psi_3) = \omega_J + \sigma_J,\ \psi_3 = \omega_J + \sigma_J + i\,^*(\omega_J + \sigma_J).$$

Lemma 4.3. *For* $\varphi \in \Lambda_a$

$$\lambda_j(\varphi) = \{2a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) + 2\{\lambda_j(\psi_3) + \mu_j^J(\overline{\varphi})\},$$

$$\lambda_J(\varphi) = \{2a_J(\overline{\varphi}) - 1\}\lambda_J(\overline{\varphi}) + 2\{\lambda_J(\psi_3) + \mu_J^J(\overline{\varphi})\}.$$

Proof. Since

$$\begin{aligned}
&\varphi = \lambda_j(\varphi) + i^*\lambda_j(\varphi) = \lambda'_{JR}(\varphi) + \lambda'_{JI}(\varphi) + i^*\{\lambda'_{JR}(\varphi) + \lambda'_{JI}(\varphi)\} \\
&= \lambda'_{JR}(\varphi) - \lambda'_{JI}(\varphi) + 2\lambda'_{JI}(\varphi) + i^*\{\lambda'_{JR}(\varphi) + \lambda'_{JI}(\varphi)\} \\
&= \lambda_J(\overline{\varphi}) + 2(\omega_j + i^*\sigma_j) + i^*\{\lambda_J(\overline{\varphi}) + 2(\omega_j + i^*\sigma_j)\} \\
&= \lambda_J(\overline{\varphi}) + 2(\omega_j + \sigma_j) + i^*\{\lambda_J(\overline{\varphi}) + 2(\omega_j + \sigma_j)\} \\
&= [a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) - \{1 - a_{Jj}(\overline{\varphi})\}i^*\lambda_j(\overline{\varphi}) + 2(\omega_j + \sigma_j)] \\
&+i^*[a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) - \{1 - a_{Jj}(\overline{\varphi})\}i^*\lambda_j(\overline{\varphi}) + 2(\omega_j + \sigma_j)] + 2\{\mu_j^J(\overline{\varphi}) + i^*\mu_j^J(\overline{\varphi})\} \\
&= \{2a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) + 2\lambda_j(\psi_3) + i^*[\{2a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) + 2\lambda_j(\psi_3)] \\
&\qquad +2\{\mu_j^J(\overline{\varphi}) + i^*\mu_j^J(\overline{\varphi})\},
\end{aligned}$$

we get

$$\lambda_j(\varphi) = \{2a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) + 2\{\lambda_j(\psi_3) + \mu_j^J(\overline{\varphi})\},$$

$$\lambda_J(\varphi) = \{2a_J(\overline{\varphi}) - 1\}\lambda_J(\overline{\varphi}) + 2\{\lambda_J(\psi_3) + \mu_J^J(\overline{\varphi})\}.$$

□

Corollary 4.1. *If* $\overline{\varphi} \in \Lambda_s + i^*\Lambda_s$ *i.e.* $a_J(\overline{\varphi}) = 1$, *then*

$$\lambda_J(\varphi) = \lambda_J(\overline{\varphi}) + 2\lambda_J(\psi_3).$$

If $a_J(\overline{\varphi}) = \frac{1}{2}$, *then* $\lambda_J(\varphi) = 2\lambda_J(\psi_3)$.

Proof. In these cases $\mu_J^J(\overline{\varphi}) = 0$ and the assertion is clear.

□

By Lemma 4.3, we have

$$\begin{aligned}
||\lambda_j(\varphi)||^2 = \{2a_{Jj}(\overline{\varphi}) - 1\}^2||\lambda_j(\overline{\varphi})||^2 + 4||\omega_j + \sigma_j + \mu_j^J(\overline{\varphi})||^2 \\
+4\{2a_{Jj}(\overline{\varphi}) - 1\} < \lambda_j(\overline{\varphi}), \omega_j + \sigma_j >,
\end{aligned}$$

and

$$\frac{1}{2}a_{Jj}(\overline{\varphi})\{1 - a_{Jj}(\overline{\varphi})\}||\varphi||^2 = ||\omega_j + \sigma_j + \mu_j^J(\overline{\varphi})||^2 + \{2a_{Jj}(\overline{\varphi}) - 1\} < \lambda_j(\overline{\varphi}), \omega_j + \sigma_j > .$$

We also have

$$\begin{aligned}
&\psi_2 = \lambda_J(\varphi) - i^*\lambda_J(\varphi) \\
&= \{2a_J(\overline{\varphi}) - 1\}\{\lambda_J(\overline{\varphi}) - i^*\lambda_J(\overline{\varphi})\} + 2[\lambda_J(\psi_3) + \mu_J^J(\overline{\varphi}) - i^*\{\lambda_J(\psi_3) + \mu_J^J(\overline{\varphi})\}] \\
&= \{2a_J(\overline{\varphi}) - 1\}\overline{\varphi} + 2[\lambda_J(\psi_3) + \mu_J^J(\overline{\varphi}) - i^*\{\lambda_J(\psi_3) + \mu_J^J(\overline{\varphi})\}].
\end{aligned}$$

Lemma 4.4. *For* $\varphi \in \Lambda_a$

$$\{2a_{Jj}(\varphi)-1\}\lambda_j(\varphi)-\{2a_J(\overline{\varphi})-1\}\lambda_j(\overline{\varphi})+2\mu_j^J(\varphi) = 2\{2a_{Jj}(\psi_3)-1\}\lambda_j(\psi_3)+4\mu_j^J(\psi_3),$$

$$\{2a_J(\varphi)-1\}\lambda_J(\varphi)-\{2a_J(\overline{\varphi})-1\}\lambda_J(\overline{\varphi})+2\mu_J^J(\varphi) = 2\{2a_J(\psi_3)-1\}\lambda_J(\psi_3)+4\mu_J^J(\psi_3).$$

Proof. By Lemma 4.3,

$$\begin{aligned}
&2\{\lambda_J(\psi_3) - i\,^*\lambda_J(\psi_3)\}\\
&= \lambda_J(\varphi) - \{2a_J(\overline{\varphi}) - 1\}\lambda_J(\overline{\varphi}) - 2\mu_J^J(\overline{\varphi}) - i\,^*[\lambda_J(\varphi) - \{2a_J(\overline{\varphi}) - 1\}\lambda_J(\overline{\varphi}) - 2\mu_J^J(\overline{\varphi})]\\
&= \lambda_J(\varphi) - i\,^*\lambda_J(\varphi) - \{2a_J(\overline{\varphi}) - 1\}\overline{\varphi} - 2\{\mu_J^J(\overline{\varphi}) - i\,^*\mu_J^J(\overline{\varphi})\}\\
&= \{2a_{Jj}(\varphi) - 1\}\{\lambda_j(\varphi) - i\,^*\lambda_j(\varphi)\} + 2\{\mu_j^J(\varphi) - i\,^*\mu_j^J(\varphi)\}\\
&\qquad\qquad -\{2a_J(\overline{\varphi}) - 1\}\{\lambda_j(\overline{\varphi}) - i^*\lambda_j(\overline{\varphi})\} - 2\{\mu_J^J(\overline{\varphi}) - i\,^*\mu_J^J(\overline{\varphi})\}\\
&= 2\{2a_{Jj}(\psi_3) - 1\}\{\lambda_j(\psi_3) - i\,^*\lambda_j(\psi_3)\} + 4\{\mu_j^J(\psi_3) + i\,^*\mu_j^J(\psi_3)\}.
\end{aligned}$$

□

Corollary 4.2.

(1) *If* $a_J(\varphi) = 1$ *and* $a_J(\overline{\varphi}) = 1,$ *then* $\lambda_J(\varphi) - \lambda_J(\overline{\varphi}) = 2\{2a_J(\psi_3) - 1\}\lambda_J(\psi_3) + 4\mu_J^J(\psi_3).$

(2) *If* $a_J(\varphi) = 1$ *and* $a_J(\overline{\varphi}) = \dfrac{1}{2},$ *then* $\lambda_J(\varphi) = 2\{2a_J(\psi_3) - 1\}\lambda_J(\psi_3) + 4\mu_J^J(\psi_3).$

(3) *If* $a_J(\varphi) = \dfrac{1}{2}$ *and* $a_J(\overline{\varphi}) = 1,$ *then* $-\lambda_J(\overline{\varphi}) = 2\{2a_J(\psi_3) - 1\}\lambda_J(\psi_3) + 4\mu_J^J(\psi_3).$

(4) *If* $a_J(\varphi) = \dfrac{1}{2}$ *and* $a_J(\overline{\varphi}) = \dfrac{1}{2},$ *then* $0 = 2\{2a_J(\psi_3) - 1\}\lambda_J(\psi_3) + 4\mu_J^J(\psi_3).$

Proposition 4.1. *If* $a_J(\varphi) = 1$ *and* $a_J(\overline{\varphi}) = \frac{1}{2}$*, then*

$$a_J(\psi_3) = 1,\ \|\psi_3\|^2 = \frac{1}{4}\|\varphi\|^2,\ \mu_J^J(\psi_3) = 0,\ \lambda_J(\psi_3) \in \Lambda_s,\ \textit{and}$$

$$0 =< \lambda_J(\psi_3), \lambda_J(\overline{\varphi}) > .$$

Proof. If $a_J(\overline{\varphi}) = \frac{1}{2}$, then $2\lambda_J(\overline{\varphi}) = \overline{\varphi} \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}$. By Lemma 4.2,

$$2\psi_3 = \varphi = \lambda_J(\varphi) + i\,^*\lambda_J(\varphi) \in \Lambda_s + i\,^*\Lambda_s,\ 2\lambda_J(\psi_3) = \lambda_J(\varphi) \in \Lambda_s.$$

We can get the statements.

□

Proposition 4.2. *If* $a_J(\varphi) = \frac{1}{2}$ *and* $a_J(\overline{\varphi}) = 1$*, then*

$$\|\psi_3\|^2 = \frac{1}{2}\|\varphi\|^2,\ a_J(\psi_3) = \frac{3}{4},\ \|\lambda_J(\psi_3)\|^2 = \frac{3}{16}\|\varphi\|^2,\ \|\mu_J^J(\psi_3)\|^2 = \frac{3}{256}\|\varphi\|^2,$$

$$-\lambda_J(\overline{\varphi}) = \lambda_J(\psi_3) + 4\mu_J^J(\psi_3),\ <\lambda_J(\overline{\varphi}), \lambda_J(\psi_3)>= -\frac{1}{4}\|\varphi\|^2,\ \textit{and}$$

$$\lim_{J\ni j\to\infty} ||\mu_j^J(\psi_3)||^2 = \frac{1}{64}||\varphi||^2.$$

Proof. By Lemma 2.1 and Lemma 4.4,

$$\begin{aligned}
&\{2a_{Jj}(\varphi) - 1\}^2||\lambda_j(\varphi)||^2 + ||\lambda_j(\overline{\varphi})||^2 + 4||\mu_j^J(\varphi)||^2 \\
&\qquad\qquad -2 < \{2a_{Jj}(\varphi) - 1\}\lambda_j(\varphi) + 2\mu_j^J(\varphi), \lambda_j(\overline{\varphi}) > \\
&= 4\{2a_{Jj}(\psi_3) - 1\}^2||\lambda_j(\psi_3)||^2 + 16||\mu_j^J(\psi_3)||^2 \\
&= 4\{2a_{Jj}(\psi_3) - 1\}^2\frac{1}{2}||\psi_3||^2 + 4\{a_J(\psi_3) + 2a_{Jj}(\psi_3) - 1 - 2a_{Jj}(\psi_3)^2\}||\psi_3||^2,
\end{aligned}$$

and

$$\begin{aligned}
\{4a_J(\psi_3) - 2\}||\psi_3||^2 &= \{2a_{Jj}(\varphi)^2 - 2a_{Jj}(\varphi) + 1\}||\varphi||^2 + 4||\mu_j^J(\varphi)||^2 \\
&\qquad -2 < \{2a_{Jj}(\varphi) - 1\}\lambda_j(\varphi) + 2\mu_j^J(\varphi), \lambda_j(\overline{\varphi}) > .
\end{aligned}$$

If $a_J(\varphi) = \frac{1}{2}$ and $a_J(\overline{\varphi}) = 1$, then $\varphi = 2\lambda_J(\varphi) \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}$ is orthogonal to $\lambda_J(\overline{\varphi}) \in \Lambda_s$, and $\mu_J^J(\varphi) = 0$. By Lemma 4.2, we have

$$4||\psi_3||^2 = ||\varphi - \{\lambda_J(\overline{\varphi}) + i\,^*\lambda_J(\overline{\varphi})\}||^2 = ||\varphi||^2 + ||\lambda_J(\overline{\varphi})||^2 + ||i\,^*\lambda_J(\overline{\varphi})||^2 = 2||\varphi||^2.$$

Since

$$\lim_{J\ni j\to\infty} ||\mu_j^J(\varphi)|| = \lim_{J\ni j\to\infty}\{a_J(\varphi) + 2a_{Jj}(\varphi) - 1 - 2a_{Jj}(\varphi)^2\}||\varphi||^2 = 0,$$

$$\begin{aligned}
\{4a_J(\psi_3) - 2\}||\psi_3||^2 &= \lim_{J\ni j\to\infty}[\{2a_{Jj}(\varphi)^2 - 2a_{Jj}(\varphi) + 1\}||\varphi||^2 + 4||\mu_j^J(\varphi)||^2 \\
&\qquad -2 < \{2a_{Jj}(\varphi) - 1\}\lambda_j(\varphi) + 2\mu_j^J(\varphi), \lambda_j(\overline{\varphi}) >] = \frac{1}{2}||\varphi||^2.
\end{aligned}$$

It follows that $a_J(\psi_3) = \frac{3}{4}$. By Corollary 4.2, $-\lambda_J(\overline{\varphi}) = \lambda_J(\psi_3) + 4\mu_J^J(\psi_3)$. From

$$\lambda_J(\psi_3) = a_{Jj}(\psi_3)\lambda_j(\psi_3) + \{1 - a_{Jj}(\psi_3)\}i\,^*\lambda_j(\psi_3) + \mu_j^J(\psi_3) - i\,^*\mu_j^J(\psi_3),$$

we get

$$||\lambda_J(\psi_3)||^2 = \lim_{J\ni j\to\infty} < \lambda_J(\psi_3), \lambda_j(\psi_3) >= \lim_{J\ni j\to\infty} a_{Jj}(\psi_3)||\lambda_j(\psi_3)||^2 = \frac{3}{16}||\varphi||^2.$$

Note that by Lemma 4.4,

$$< \lambda_J(\overline{\varphi}), \lambda_J(\psi_3) >= \lim_{J\ni j\to\infty} < \lambda_J(\overline{\varphi}), \lambda_j(\psi_3) >$$

$$= \lim_{J\ni j\to\infty} < \lambda_J(\overline{\varphi}) - \lambda_j(\overline{\varphi}) + \lambda_j(\overline{\varphi}), \lambda_j(\psi_3) >= \lim_{J\ni j\to\infty} < \lambda_j(\overline{\varphi}), \lambda_j(\psi_3) >$$

$$= \lim_{J\ni j\to\infty} < \{2a_{Jj}(\varphi)-1\}\lambda_j(\varphi)+2\mu_j^J(\varphi)-2\{2a_{Jj}(\psi_3)-1\}\lambda_j(\psi_3)-4\mu_j^J(\psi_3), \lambda_j(\psi_3) >$$

$$= \lim_{J\ni j\to\infty} -2\{2a_{Jj}(\psi_3) - 1\}||\lambda_j(\psi_3)||^2 = -2\{2a_J(\psi_3) - 1\}\frac{1}{2}||\psi_3||^2 = -\frac{1}{4}||\varphi||^2.$$

By Corollary 4.2,

$$16||\mu_J^J(\psi_3)||^2 = ||-\lambda_J(\overline{\varphi})-\lambda_J(\psi_3)||^2 = ||\lambda_J(\overline{\varphi})||^2+||\lambda_J(\psi_3)||^2+2 < \lambda_J(\overline{\varphi}), \lambda_J(\psi_3) >$$

$$= (\frac{1}{2} + \frac{3}{16} - \frac{1}{2})||\varphi||^2 = \frac{3}{16}||\varphi||^2,$$

and

$$\lim_{J\ni j\to\infty} ||\mu_j^J(\psi_3)||^2 = \frac{1}{4}\{2a_J(\psi_3) - 1\}\{1 - a_J(\psi_3)\}||\psi_3||^2 = \frac{1}{64}||\varphi||^2.$$

□

Let us take an orthogonal decomposition of $\lambda'_J(\varphi)$:

$\lambda'_J(\varphi) = \gamma_j + i\,^*\delta_j$, where $\gamma_j,\ \delta_j \in \Lambda_j$.

Proposition 4.3. *If* $a_J(\varphi) = \frac{1}{2}$ *and* $a_J(\overline{\varphi}) = 1,$ *then*

$$\lim_{J\ni j\to\infty} \|\omega_j\|^2 = \frac{3}{16}\|\varphi\|^2,\ \lim_{J\ni j\to\infty} \|\sigma_j\|^2 = \frac{1}{16}\|\varphi\|^2,$$

$$\lim_{J\ni j\to\infty} \|\gamma_j\|^2 = \lim_{J\ni j\to\infty} \|\delta_j\|^2 = \frac{1}{4}\|\varphi\|^2, < \lambda_J(\overline{\varphi}), \sigma_J >= 0,\ \textit{and}$$

$$< \lambda_J(\overline{\varphi}), \omega_J >= -\frac{1}{4}||\varphi||^2.$$

Proof. If $a'_J(\varphi) = a_J(\overline{\varphi}) = 1$, then $\lambda'_J(\varphi) \in \Lambda'_s$. Hence we have

$$0 =< \lambda'_J(\varphi), i\,^*\lambda'_J(\varphi) >=< \gamma_j + i^*\delta_j, i\,^*\gamma_j + \delta_j >= 2 < \gamma_j, \delta_j > .$$

From

$$\lambda'_J(\varphi) = \lambda'_{JR}(\varphi) - \lambda'_{JI}(\varphi) + 2\lambda'_{JI}(\varphi) = \lambda_J(\overline{\varphi}) + 2(\omega_j + i\,^*\sigma_j),$$

we get

$$\gamma_j = a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}) + 2\omega_j,\ \delta_j = \{a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}) + 2\sigma_j.$$

From

$$\lambda_j(\varphi) + i\,{}^*\lambda_j(\varphi) = \varphi = \lambda'_J(\varphi) + i\,{}^*\lambda'_J(\varphi) = (\gamma_j + \delta_j) + i\,{}^*(\gamma_j + \delta_j),$$

we get $\lambda_j(\varphi) = \gamma_j + \delta_j$. It follows that

$$< \lambda_j(\varphi), \lambda'_J(\varphi) >=< \gamma_j + \delta_j, \gamma_j + i\,{}^*\delta_j >= ||\gamma_j||^2,$$

$$\lim_{J\ni j\to\infty} < \lambda_j(\varphi), \lambda'_J(\varphi) >=< \lambda_J(\varphi), \lambda'_J(\varphi) >=< \frac{1}{2}\varphi, \lambda'_J(\varphi) >$$
$$= \frac{1}{2} < \lambda'_J(\varphi) + i\,{}^*\lambda'_J(\varphi), \lambda'_J(\varphi) >= \frac{1}{2}||\lambda'_J(\varphi)||^2 = \frac{1}{4}||\varphi||^2.$$

Therefore

$$\lim_{J\ni j\to\infty} ||\gamma_j||^2 = \frac{1}{4}||\varphi||^2,\ \lim_{J\ni j\to\infty} ||\delta_j||^2 = \lim_{J\ni j\to\infty}\{||\lambda_j(\varphi)||^2 - ||\gamma_j||^2\} = \frac{1}{4}||\varphi||^2,$$

$$\lim_{J\ni j\to\infty} ||\sigma_j||^2 = \frac{1}{4}\lim_{J\ni j\to\infty} ||\delta_j - \{a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) - \mu_j^J(\overline{\varphi})||^2 = \frac{1}{16}||\varphi||^2, \text{ and}$$

$$\lim_{J\ni j\to\infty} ||\omega_j||^2 = \lim_{J\ni j\to\infty}\{||\lambda_j(\psi_3)||^2 - ||\sigma_j||^2\} = (\frac{1}{4} - \frac{1}{16})||\varphi||^2 = \frac{3}{16}||\varphi||^2.$$

From

$$0 =< \gamma_j, \delta_j >=< a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}) + 2\omega_j, \{a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}) + 2\sigma_j >$$
$$=< a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}) + 2\omega_j, \{a_{Jj}(\overline{\varphi}) - 1\}\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}) >$$
$$+2 < a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}), \sigma_j >,$$

we have $< \lambda_J(\overline{\varphi}), \sigma_J >= 0$. Since

$$||\gamma_j||^2 = ||a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) + \mu_j^J(\overline{\varphi}) + 2\omega_j||^2$$
$$= a_{Jj}(\overline{\varphi})^2||\lambda_j(\overline{\varphi})||^2 + 4||\omega_j||^2 + 4a_{Jj}(\overline{\varphi}) < \lambda_J(\overline{\varphi}), \omega_j >$$
$$+ < a_{Jj}(\overline{\varphi})\lambda_j(\overline{\varphi}) + 2\omega_j, \mu_j^J(\overline{\varphi}) >,$$

we have

$$< \lambda_J(\overline{\varphi}), \omega_J >= \frac{1}{4}(\frac{1}{4} - \frac{1}{2} - \frac{3}{4})||\varphi||^2 = -\frac{1}{4}||\varphi||^2.$$

□

Proposition 4.4. *For $\varphi \in \Omega_a$, we have the orthogonal decompositions:*

$$\lambda'_J(\varphi) = \chi_1 - i^*\chi_1 + \theta,$$

where $\chi_1 \in \Lambda_s$, $\theta \in \Lambda_{bw} \cap i^*\Lambda_{bw}$. *If* $a_J(\varphi) = \frac{1}{2}$ *and* $a_J(\overline{\varphi}) = 1$, *then*

$$\omega_J = -\frac{1}{2}\lambda_J(\overline{\varphi}) + \frac{1}{2}\chi_1 + \theta_1, \quad \sigma_J = -\frac{1}{2}\chi_1 + \frac{1}{2}\lambda_J(\varphi) - \theta_1,$$

where $\theta_1 \in \Lambda_{bw} \cap i^*\Lambda_{bw}$ *and* $\theta_1 - i^*\theta_1 = (\theta - i^*\theta)/4$

Proof. For $\varphi \in \Omega_a$, take an orthogonal decomposition:

$$\lambda'_J(\varphi) = \chi_1 + i^*\chi_2 + \theta, \text{ where } \chi_1, \chi_2 \in \Lambda_s \text{ and } \theta \in \Lambda_{bw} \cap i^*\Lambda_{bw}.$$

From

$$\varphi = \lambda'_J(\varphi) + i^*\lambda'_J(\varphi) = \chi_1 + \chi_2 + i^*(\chi_1 + \chi_2) + \theta + i^*\theta \in \Lambda_{bw} \cap i^*\Lambda_{bw},$$

we have $\chi_2 = -\chi_1$ and $\lambda'_J(\varphi) = \chi_1 - i^*\chi_1 + \theta$. For $\nu \in i^*\Lambda_s$, we have

$$< \omega_J, \nu >= \lim_{J \ni j \to \infty} (< \omega_j, \nu_j > + < \omega_j, \nu - \nu_j >) = 0,$$

where $\nu_j \in i^*\Lambda_j$ and $\lim_{J \ni j \to \infty} \|\nu - \nu_j\| = 0$. Similarly we have $< \sigma_J, \nu >= 0$. Hence we can set

$$\omega_J = \chi_3 + \theta_1, \ \sigma_J = \chi_4 + \theta_2, \text{ where } \chi_3, \ \chi_4 \in \Lambda_s \text{ and } \theta_1, \theta_2 \in \Lambda_{bw} \cap i^*\Lambda_{bw}.$$

Then $\omega_J + \sigma_J = \chi_3 + \chi_4 + \theta_1 + \theta_2 = \lambda_J(\psi_3)$. When $a_J(\overline{\varphi}) = 1$, we have $\mu^J_J(\overline{\varphi}) = 0$ and by Lemma 4.3, $2\lambda_J(\psi_3) = 2(\omega_J + \sigma_J) = \lambda_J(\varphi) - \lambda_J(\overline{\varphi})$. Since $\lambda_J(\varphi) \in \Lambda_{bw} \cap i^*\Lambda_{bw}$ and $\lambda_J(\overline{\varphi}) \in \Lambda_s$, we have $\chi_3 + \chi_4 = -\frac{1}{2}\lambda_J(\overline{\varphi})$ and $\theta_1 + \theta_2 = \frac{1}{2}\lambda_J(\varphi)$. Note that by Lemma 4.2,

$$\begin{aligned}
&\lambda'_J(\varphi) - i^*\lambda'_J(\varphi) = 2(\chi_1 - i^*\chi_1) + \theta - i^*\theta \\
&= \lambda_J(\overline{\varphi}) + 2\{\omega_J + i^*\sigma_J\} - i^*(\lambda_J(\overline{\varphi}) + 2\{\omega_J + i^*\sigma_J)\} \\
&= \lambda_J(\overline{\varphi}) - i^*\lambda_J(\overline{\varphi}) + 2\{\omega_J - \sigma_J - i^*(\omega_J - \sigma_J)\} \\
&= \lambda_J(\varphi) - 2\lambda_J(\psi_3) - i^*\{\lambda_J(\varphi) - 2\lambda_J(\psi_3)\} + 2\{\omega_J - \sigma_J - i^*(\omega_J - \sigma_J)\} \\
&= \psi_2 - 4\sigma_J + 4i^*\sigma_J = \psi_2 - 4(\chi_4 - i^*\chi_4 + \theta_2 - i^*\theta_2).
\end{aligned}$$

Since $a_J(\varphi) = \frac{1}{2}$, we have $\psi_2 = 0$. Therefore we get $\chi_4 = -\chi_1/2$, $-\theta_2 + i^*\theta_2 = (\theta - i^*\theta)/4$, $\chi_3 = \frac{1}{2}\{-\lambda_J(\overline{\varphi}) + \chi_1\}$, and $\theta_1 - i^*\theta_1 = (\theta - i^*\theta)/4$. $\square$

Lemma 4.5. *If $a_J(\varphi) = \frac{1}{2}$ and $a_J(\overline{\varphi}) = 1$, then*

$$< \chi_1, \lambda_J(\overline{\varphi}) >= 0,\ \frac{1}{4}||\chi_1||^2 + ||\theta_1||^2 = ||\sigma_J||^2,$$

$$||\omega_J||^2 + < \omega_J, i^*\sigma_J >= \frac{3}{16}||\varphi||^2,\ \textit{and}\ \ ||\sigma_J||^2 + < \sigma_J, i^*\omega_J >= \frac{1}{16}||\varphi||^2.$$

Proof. By Proposition 4.3, we have

$$-\frac{1}{4}||\varphi||^2 =< \omega_J, \lambda_J(\overline{\varphi}) >=< -\frac{1}{2}\lambda_J(\overline{\varphi}) + \frac{1}{2}\chi_1 + \theta_1, \lambda_J(\overline{\varphi}) >$$

$$= -\frac{1}{4}||\varphi||^2 + < \frac{1}{2}\chi_1 + \theta_1, \lambda_J(\overline{\varphi}) > .$$

Since $\lambda'_J(\varphi) \in \overline{\Lambda}_s$, we get

$$0 =< \frac{1}{2}\chi_1 + \theta_1, \lambda_J(\overline{\varphi}) >= \frac{1}{2} < \chi_1, \lambda_J(\overline{\varphi}) > + < \theta_1, \overline{\lambda'_J(\varphi)} >= \frac{1}{2} < \chi_1, \lambda_J(\overline{\varphi}) > .$$

By Proposition 4.4, we have

$$||\omega_J||^2 = \frac{1}{4}||\lambda_J(\overline{\varphi})||^2 + ||\frac{1}{2}\chi_1 + \theta_1||^2 = \frac{1}{8}||\varphi||^2 + \frac{1}{4}||\chi_1||^2 + ||\theta_1||^2.$$

Note that

$$||\omega_J||^2 + < \omega_J, i^*\sigma_J >=< \omega_J, \omega_J + i^*\sigma_J >=< \omega_J, \lambda'_{JI} >= \lim_{J \ni j \to \infty} < \omega_j, \lambda'_{JI} >$$

$$= \lim_{J \ni j \to \infty} < \omega_j, \omega_j + i^*\sigma_j >= \lim_{J \ni j \to \infty} < \omega_j, \omega_j >= \frac{3}{16}||\varphi||^2 \ \text{ and}$$

$$||\sigma_J||^2 + < \sigma_J, i^*\omega_J >=< \sigma_J, i^*\omega_J + \sigma_J >=< \sigma_J, i^*\lambda'_{JI} >$$

$$= \lim_{J \ni j \to \infty} < \sigma_j, i^*\omega_j + \sigma_j >= \lim_{J \ni j \to \infty} < \sigma_j, \sigma_j >= \frac{1}{16}||\varphi||^2.$$

It follows that

$$||\omega_J||^2 = ||\sigma_J||^2 + \frac{1}{8}||\varphi||^2 \text{ and } \frac{1}{4}||\chi_1||^2 + ||\theta_1||^2 = ||\sigma_J||^2.$$

□

Proposition 4.5. *If $a_J(\varphi) = \frac{1}{2}$ and $a_J(\overline{\varphi}) = 1$, then*

$$\frac{1}{2}\|\chi_1\|^2 + < \theta_1, \theta >= \frac{1}{8}\|\varphi\|^2,\ < \theta, i^*\theta >= 4 < \theta_1, i^*\theta >= 2\|\chi_1\|^2,\ \textit{and}$$

$$\|\theta - i^*\theta\|^2 = \|\varphi\|^2 - 8\|\chi_1\|^2.$$

Proof. By Proposition 4.2,

$$\lambda_J(\overline{\varphi}) = -\lambda_J(\psi_3) - 4\mu_J^J(\psi_3),\ \lambda_J(\psi_3) = \omega_J + \sigma_J,$$

and by Corollary 4.1,

$$\lambda_J(\varphi) = \lambda_J(\overline{\varphi}) + 2\lambda_J(\psi_3) = \lambda_J(\psi_3) - 4\mu_J^J(\psi_3).$$

By Lemmas 4.2, 4.5 and Propositions 4.3, 4.4,

$$< \omega_J, \lambda'_J(\varphi) >=< \omega_J, \lambda_J(\overline{\varphi}) + 2(\omega_j + i\,^*\sigma_j) >= (-\frac{1}{4} + \frac{3}{8})\|\varphi\|^2 = \frac{1}{8}\|\varphi\|^2$$

$$=< -\frac{1}{2}\lambda_J(\overline{\varphi}) + \frac{1}{2}\chi_1 + \theta_1, \chi_1 - i\,^*\chi_1 + \theta >= \frac{1}{2}\|\chi_1\|^2 + < \theta_1, \theta > .$$

Since $i\,^*\lambda'_J(\varphi) \in i\,^*\overline{\Lambda_s}$, we have, by Proposition 4.4,

$$0 =< \omega_J, i\,^*\lambda'_J(\varphi) >=< -\frac{1}{2}\lambda_J(\overline{\varphi}) + \frac{1}{2}\chi_1 + \theta_1, i\,^*\chi_1 - \chi_1 + i\,^*\theta >$$

$$= -\frac{1}{2}\|\chi_1\|^2 + < \theta_1, i\,^*\theta > .$$

It follows that

$$< \theta_1, \theta - i\,^*\theta >= \frac{1}{8}\|\varphi\|^2 - \|\chi_1\|^2 = \frac{1}{2} < \theta_1 - i\,^*\theta_1, \theta - i\,^*\theta >= \frac{1}{8}\|\theta - i\,^*\theta\|^2,$$

and

$$0 =< \lambda'_J(\varphi), i\,^*\lambda'_J(\varphi) >=< \chi_1 - i\,^*\chi_1 + \theta, i\,^*\chi_1 - \chi_1 + i\,^*\theta >$$

$$= -2\|\chi_1\|^2 + < \theta, i\,^*\theta > .$$

□

5. Linear mappings between Ω_a and $\Omega_{\bar{a}}$

We can construct a behavior space if the dimensions of Ω_a and $\Omega_{\bar{a}}$ are same (cf. [3]). So we consider certain linear mappings between Ω_a and $\Omega_{\bar{a}}$. Let $\Omega_{a-} = \Omega_{a.5} \cap (\overline{\Lambda}_s + i\,^*\overline{\Lambda}_s)$, $\{\Omega_{\bar{a}-} = \Omega_{\bar{a}.5} \cap (\overline{\Lambda}_s + i\,^*\overline{\Lambda}_s)\}$, $\Omega_{a+} = \Omega_{a.5} \cap \Omega_{a-}^{\perp}$, $\{\Omega_{\bar{a}+} = \Omega_{\bar{a}.5} \cap \Omega_{\bar{a}-}^{\perp}\}$, $\Omega_{a.5} = \Omega_{a+} + \Omega_{a-}$ and $\{\Omega_{\bar{a}.5} = \Omega_{\bar{a}+} + \Omega_{\bar{a}-}\}$. We define a real linear mapping F from $\Omega_a = \Omega_{a0} + \Omega_{a+} + \Omega_{a-}$ to $\Omega_{\bar{a}} = \Omega_{\bar{a}0} + \Omega_{\bar{a}+} + \Omega_{\bar{a}-}$ as follows; for $\varphi \in \Omega_a$, take the orthogonal decomposition $\varphi = \varphi_0 + \varphi_+ + \varphi_-$ ($\varphi_0 \in \Omega_{a0}, \varphi_+ \in \Omega_{a+}$ and $\varphi_- \in \Omega_{a-}$), and put

$$F(\varphi) = \lambda_J(\varphi_0) - i\,^*\lambda_J(\varphi_0) + P(\overline{\varphi_+}) + P(\lambda'_J(\varphi_-)) - i\,^*P(\lambda'_J(\varphi_-)),$$

where $P(\overline{\varphi_+})$ and $P(\lambda'_J(\varphi_-))$ are the orthogonal projections of $\overline{\varphi_+}$ and $\lambda'_J(\varphi_-)$ to $\Lambda_{bw} \cap i\,^*\Lambda_{bw}$ respectively. Since $\lambda_J(\varphi_0) \in \Lambda_{bw}$, we have $i\,^*\lambda_J(\varphi_0) = \varphi_0 - \lambda_J(\varphi_0) \in \Lambda_{bw},\ \lambda_J(\varphi_0) \in i\,^*\Lambda_{bw}$ and $\lambda_J(\varphi_0) \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}$. Set $\lambda_J(\varphi_0) = \theta_0$ and then $\theta_0 - i\,^*\theta_0 \in \Omega_{\bar{a}}$. Let us take the orthogonal decomposition $\overline{\varphi_+} = \varphi_2 + i\,^*\varphi_3 + \theta_+$, where $\varphi_2,\ \varphi_3 \in \Lambda_s$ and $\theta_+ \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}$. Then $\varphi_2 + i\,^*\varphi_3 + \theta_+ = -i\,^*\varphi_2 - \varphi_3 - i^*\theta_+$ and $\varphi_3 = -\varphi_2,\ P(\overline{\varphi_+}) = \theta_+ = -i\,^*\theta_+ \in \Omega_{\bar{a}}$. Hence

$$\overline{\varphi_+} = \varphi_2 - i^*\varphi_2 + \theta_+,\ \text{where } \varphi_2 \in \Lambda_s \text{ and } \theta_+ \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}.$$

Let $\varphi_5 = \varphi_2 - i\,^*\varphi_2$, then $\varphi_5 = \lambda_n(\varphi_5) - i\,^*\lambda_n(\varphi_5),\ a_J(\varphi_5) = 1$. From the orthogonal decomposition $\lambda'_J(\varphi_-) = \chi_- + i^*\chi'_- + \theta_-$, where $\chi_-,\ \chi'_- \in \Lambda_s$ and $\theta_- \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}$, we have $\theta_- = P(\lambda'_J(\varphi_-)),\ \theta_- - i\,^*\theta_- \in \Omega_{\bar{a}}$, and

$$\varphi_- = \lambda'_J(\varphi_-) + i^*\lambda'_J(\varphi_-) = \chi_- + \chi'_- + i\,^*(\chi_- + \chi'_-) + \theta_- + i\,^*\theta_- \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}.$$

Hence $\chi'_- = -\chi_-,\ \varphi_- = \theta_- + i\,^*\theta_-$, and $\lambda'_J(\varphi_-) = \chi_- - i\,^*\chi_- + \theta_-$. Therefore

$$F(\varphi) = \theta_0 - i\,^*\theta_0 + \theta_+ + \theta_- - i\,^*\theta_-.$$

Take an orthogonal decomposition :

$$\theta_* = \sum_{j=1}^{6} \theta_{*j}, \text{where } \theta_{*1} \in \Omega_{a0},\ \theta_{*2} \in \Omega_{a+},\ \theta_{*3} \in \Omega_{a-},\ \theta_{*4} \in \Omega_{\bar{a}0},\ \theta_{*5} \in \Omega_{\bar{a}+},$$

$\theta_{*6} \in \Omega_{\bar{a}-}$ and $*$ denotes 0 or $+$ or $-$. From

$$\varphi_0 = \theta_0 + i\,^*\theta_0 = \sum_{j=1}^{6}(\theta_{0j} + i\,^*\theta_{0j}) = 2\sum_{j=1}^{3}\theta_{0j} \in \Omega_{a0},$$

we have $\theta_{01} = \frac{1}{2}\varphi_0,\ \theta_{02} = \theta_{03} = 0$. Since θ_+ is anti-holomorphic, $\theta_{+1} = \theta_{+2} = \theta_{+3} = 0$. From

$$\varphi_+ = \overline{\varphi_2} + i\,^*\overline{\varphi_2} + \sum_{i=4}^{6}\overline{\theta_{+i}},$$

we get

$$\|\varphi_+\|^2 = 2\|\varphi_2\|^2 + \sum_{i=4}^{6}\|\theta_{+i}\|^2.$$

On the other hand, we have the following representation:

$$\overline{\theta_{+6}} = \varphi_6 + i\,^*\varphi_6, \text{where } \varphi_6 \in \Lambda_s,$$

and

$$\varphi_+ - \varphi_6 - i\,^*\varphi_6 = \overline{\varphi_2} + i\,^*\overline{\varphi_2} + \sum_{i=4}^{5}\overline{\theta_{+i}}.$$

Hence,

$$\|\varphi_+\|^2 + 2\|\varphi_6\|^2 = 2\|\varphi_2\|^2 + \sum_{i=4}^{5} \|\theta_{+i}\|^2.$$

This shows $\theta_{+6} = 0$. From

$$\varphi_- = \theta_- + i\,^*\theta_- = \sum_{j=1}^{6}(\theta_{-j} + i\,^*\theta_{-j}) = 2\sum_{j=1}^{3} \theta_{-j} \in \Omega_{a-},$$

we have $\theta_{-3} = \frac{1}{2}\varphi_-,\ \theta_{-1} = \theta_{-2} = 0$. By Proposition 3.1, we have

Proposition 5.1.

$$F(\varphi) = 2\theta_{04} + \theta_{+4} + \theta_{+5} + 2\sum_{j=4}^{6} \theta_{-j}.$$

Proposition 5.2. *F is injective on Ω_{a0} and Ω_{a+} respectively.*

Proof. When $\lambda_J(\varphi_0) - i\,^*\lambda_J(\varphi_0) = 0$, we have

$$\lambda_J(\varphi_0) = i\,^*\lambda_J(\varphi_0) \text{ and } \varphi_0 = \lambda_J(\varphi_0) + i\,^*\lambda_J(\varphi_0) = 2\lambda_J(\varphi_0).$$

Hence $\varphi_0 \in \Omega_{a.5}$ and $\varphi_0 = 0$. When $\theta_+ = 0$, by definition $\varphi_+ \in \Omega_- \cap \Omega_{a+}$. Hence $\varphi_+ = 0$.

□

We define a real linear mapping G from $\Omega_{\bar{a}}$ to Ω_a as follows; for $\bar{\psi} \in \Omega_{\bar{a}}$ take the orthogonal decomposition $\bar{\psi} = \bar{\psi}_0 + \bar{\psi}_+ + \bar{\psi}_-$, where $\bar{\psi}_0 \in \Omega_{\bar{a}0}, \bar{\psi}_+ \in \Omega_{\bar{a}+}, \bar{\psi}_- \in \Omega_{\bar{a}-}$, and put

$$G(\bar{\psi}) = \lambda_J(\bar{\psi}_0) + i\,^*\lambda_J(\bar{\psi}_0) + P(\bar{\psi}_+) + P(\lambda'_J(\bar{\psi}_-)) + i\,^*P(\lambda'_J(\bar{\psi}_-)),$$

where $P(\overline{\bar{\psi}_+})$ and $P(\lambda'_J(\bar{\psi}_-))$ are the orthogonal projection of $\overline{\bar{\psi}_+}$ and $\lambda'_J(\bar{\psi}_-)$ to $\Lambda_{bw} \cap i\,^*\Lambda_{bw}$ respectively. Since $\lambda_J(\bar{\psi}_0) \in \Lambda_{bw}$, we have $i\,^*\lambda_J(\bar{\psi}_0) = -\bar{\psi}_0 + \lambda_J(\bar{\psi}_0) \in \Lambda_{bw},\ \lambda_J(\bar{\psi}_0) \in i\,^*\Lambda_{bw}$ and $\lambda_J(\bar{\psi}_0) \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}$. Set $\lambda_J(\bar{\psi}_0) = \eta_0$ and then $\eta_0 + i\,^*\eta_0 \in \Omega_a$. Let us take the orthogonal decomposition:

$$\overline{\bar{\psi}_+} = \bar{\psi}_2 + i\,^*\bar{\psi}_3 + \eta_+, \text{ where } \bar{\psi}_2,\ \bar{\psi}_3 \in \Lambda_s \text{ and } \eta_+ \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}.$$

Then $\bar{\psi}_2 + i\,^*\bar{\psi}_3 + \eta_+ = i\,^*\bar{\psi}_2 + \bar{\psi}_3 + i^*\eta_+$ and $\bar{\psi}_3 = \bar{\psi}_2,\ P(\overline{\bar{\psi}_+}) = \eta_+ = i\,^*\eta_+ \in \Omega_a$. Hence

$$\overline{\bar{\psi}_+} = \bar{\psi}_2 + i^*\bar{\psi}_2 + \eta_+, \text{ where } \bar{\psi}_2 \in \Lambda_s \text{ and } \eta_+ \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}.$$

Let $\bar{\psi}_5 = \bar{\psi}_2 + i\,^*\bar{\psi}_2$, then $\bar{\psi}_5 = \lambda_n(\bar{\psi}_5) + i\,^*\lambda_n(\bar{\psi}_5),\ a_J(\bar{\psi}_5) = 1$. From the orthogonal decomposition $\lambda'_J(\bar{\psi}_-) = \kappa_- + i^*\kappa'_- + \eta_-$, where $\kappa_-,\ \kappa'_- \in \Lambda_s$, and $\eta_- \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}$, we have $\eta_- = P(\lambda'_J(\bar{\psi}_-)),\ \eta_- + i\,^*\eta_- \in \Omega_a$, and

$$\bar{\psi}_- = \lambda'_J(\bar{\psi}_-) - i^*\lambda'_J(\bar{\psi}_-) = \kappa_- - \kappa'_- + i\,^*(\kappa_- - \kappa'_-) + \eta_- - i\,^*\eta_- \in \Lambda_{bw} \cap i\,^*\Lambda_{bw}.$$

Hence $\kappa'_- = \kappa_-,\ \bar{\psi}_- = \eta_- - i\,^*\eta_-$, and

$$\lambda'_J(\bar{\psi}_-) = \kappa_- + i\,^*\kappa_- + \eta_-,\ G(\bar{\psi}) = \eta_0 + i\,^*\eta_0 + \eta_+ + \eta_- + i\,^*\eta_-.$$

Take an orthogonal decomposition :

$$\eta_* = \sum_{j=1}^{6} \eta_{*j},\ \text{where } \eta_{*1} \in \Omega_{a0},\ \eta_{*2} \in \Omega_{a+},\ \eta_{*3} \in \Omega_{a-},\ \eta_{*4} \in \Omega_{\bar{a}0},\ \eta_{*5} \in \Omega_{\bar{a}+},$$

$\eta_{*6} \in \Omega_{\bar{a}-}$ and $*$ denotes 0 or $+$ or $-$. Then by the same way as in $F(\varphi)$ we have

Proposition 5.3.

$$G(\varphi) = 2\eta_{01} + \eta_{+1} + \eta_{+2} + 2\sum_{j=1}^{3} \eta_{-j}.$$

Proposition 5.4. *G is injective on $\Omega_{\bar{a}0}$ and $\Omega_{\bar{a}+}$ respectively.*

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