Applications of L–Mosaic Map to Spatial Analysis of Urban Map

By

Masao FURUYAMA

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Abstract

In regional science, many researches have been focused on the study of point facilities (for example, hospital, school, and etc.) but network facilities (for examrle, road, railway) have not been investigated in depth. Hence, we are concerned with network facilities in this paper, especially the distribution of crossing numbers, the distribution of the length of segments, crossing angles and the area of cells are analyzed by applying a random line mosaid map. A random line mosaic map is shown to be useful for discribing intersecting road, river, railway, electric wire and administrative boundary line in an urban area. An empilical study is pursued . in Japanese cities.

1. Introduction

The purpose of this paper is to study characteristics of L-masaic and apply it to the analysis of several urban areas.

1-1 Descriptive model for network maps by use of L-mosaic map.

Hitherto, some important advances have occured in the approach of point process in distribution model of urban facilities. These results have any relation to descriptive model developed in the area of the forest ecology, urban geography, regional science. Nevertheless, the urban map of the distributed random line segments have not been fully discussed in urban geography and so we will study here some urban maps by the approach of random line process.

First, we shall deal with the random line mosaic map and describe its many characteristics. Second, we shall compare it with some Japanese urban maps empirically.

1-2 Mosaic-Maps

Mosaic-map has been discussed in mathematical ecology (for instance, see (4)) and two types of mosaic maps are often referred to. One is a random set mosaic (see fig. 1) and the other is random line masaic. (see fig. 2) These are both used to analyze ecological maps. Besides these two maps, we propose a random covering mosaic map.

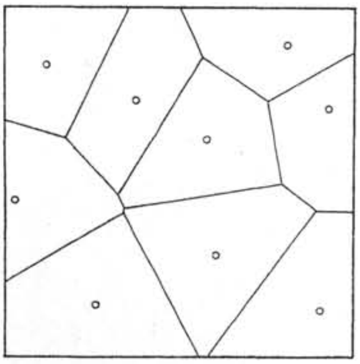
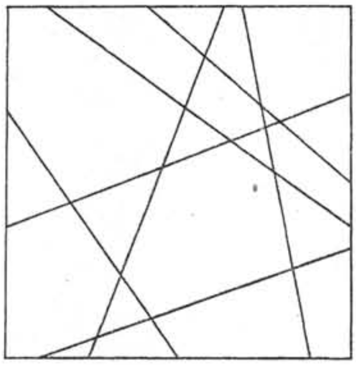
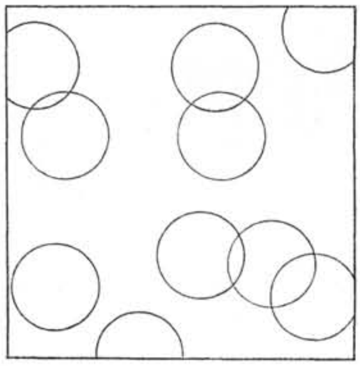




Fig. 1 Random set mosaic Fig. 2 Random line mosaic Fig. 3 Random covering mosaic

(see fig. 3) For the sake of brevity, a random set mosaic is called S-mosaic, a random line mosaic is called L-mosaic and a random covering mosaic is called C-mosaic hereafter. In this paper, our main concern is with L-mosaic and its characteristics. L-mosaic map is examined in such fields, as mathematical ecology, geometrical propability and crystal analysis. In this paper, we shall show its application to urban geography.

1-3 Description model by use of a random mosaic map.

One may consider that ecological maps or crystal maps should not be applied to an urban map because urban areas are artificially built. However, we find a justification for its use as follows; urban area has been planned considering any local optimum but to accumulate local optimum plans is not necessarily lead to groval optimum plan of urban area and sometimes to random distribution of public facilities. This hypothesis will be supported by the finding that several theoretical estimates by L-mosaic map are fitted to the data collected in Japanese many cities.

1-4 Role of descriptive models for urban planning

The reason why we wish to use model recognition of the relation between descriptive model and urban planing must be declared, after that, explain that descriptive model is useful tool for urban analysis.

a. To understand clearly; an urban pattern is so complex that a lot of indexes may be necessary to describe it. Too much information is, however, not always useful for planners. Rather, a simple index which represents a grobal urban pattern would be practical to them.

b. To obtain a theoretical basis; the rule of urban distribution is sometime got by reduction way and so it is experience rule but theoretical model is more operational and generally useful for planning.

c. To obtain relationship between indexes of urban morphology; by the empirical method, morphological unit of urban planning will be estimated but to get relation betweed indexes of pattern is very difficult by that way. If we apply the L-mosaic map, we will be able to get the relationship easily.

d. To get module of urban pattern; if we can get the descriptive model to satisfy the above three conditions **1, 2, 3,** we will be able to construct the module of morphological unit and so very useful for our urban planning because we can clear up the indexes and their relationships of comprehensive urban morphological pattern.

2. L-Mosaic Map

L-mosaic map is employed for analizing the distribution of random lines. We shall show first definition of this map. Second its applications will be shown. A straight line L in the plane is determined by its polar co-ordinates (p, θ). (see fig. 4) That is, the equation of line L is written as

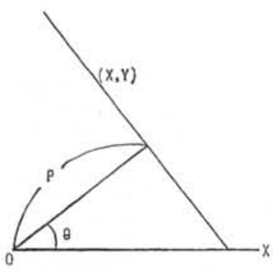


Fig. 4 Definition of random line.

$$x \cos \theta + y \sin \theta = p \tag{1}$$

generally the measure of set G of straight lines is given by equation

$$m(G) = \int_G f(p, \theta)\, dp d\theta. \tag{2}$$

In order to be a reasonable measure of random lines, the value of this integral should be invariant under affine transformation. By affine transformation, line L becomes

$$(a + x' \cos \alpha - y' \sin \alpha) \cos \theta + (b + x' \sin \alpha + y' \cos \alpha) \sin \theta = p \tag{3}$$

Rearranging this, we obtain

$$x' \cos (\theta - \alpha) + y' \sin (\theta - a) = p - a \cos \theta - b \sin \theta \tag{4}$$

Compared equ. (1) to (4), the relation

$$\theta' = \theta - \alpha,\ p' = p - a \cos \theta - b \sin \theta$$

holds.

If $m(G)$ is invariant, we have

$$\int_{X'} f(p', \theta') dp' d\theta' = \int f(p, \theta) dp d\theta \tag{5}$$

On the other hand, since Jacobian transformation is given by

$$\frac{\partial(p', \theta')}{\partial(p, \theta)} = \begin{vmatrix} 1 & a\sin\theta - b\cos\theta \\ 0 & 1 \end{vmatrix} = 1 ,$$

We obtain

$$\int_{X'} f(p', \theta')dp'd\theta' = \int_X f(p', \theta')dpd\theta \ . \tag{6}$$

From equation (5) and (6)

$$\int_X f(p, \theta)dpd\theta = \int_X f(p', \theta')dpd\theta$$

since this equality should hold for any set $G(m)$, the equation

$$f(p, \theta) = f(p', \theta') \tag{7}$$

holds. This implies $f(p, \theta) = constant$. We fix this constant to be unity for convenience. Hence we finally obtain

$$m(G) = \int dpd\theta \ . \tag{8}$$

It is noted that many useful properties of random lines are obtained by use of integral geometry[5),6)]. Concerning its application, the distribution of mixed many species of trees[4)], and mineral crystal are analyzed[10)]. For instance, see (4), (6), (13) among others.

3. Morphological Index of L-Mosaic

3-1 Four aspects of morphological indexes.

When we wish to graspe characteristics of a network pattern by a morphological index, what indexes should we choose? For example, if we observe Horton's rule of river network, Fullman's formula of geological crystals, Kansky's definition of transportation network analysis, we find the following four indexes: Number of node, length of link, area of cell, angle of crossing lines. These four indexes would also be useful for analysis of urban maps. On the other hand, as example of another index, we know 'random mingling' expressing the relation of adjacentment, see (4).

3-2 Probability of random line intersecting each other in a domain

a. Measure of straight lines which intersect a curve

Let c be a fixed curve composed of a finite number of arcs with tangent at every point. When c has a finite length L, the measure of all straight lines which intersect c can be obtained

$$\int n dpd\theta = 2L \tag{9}$$

where n is the number of crossing that straight line intersect c. (see fig. 5)

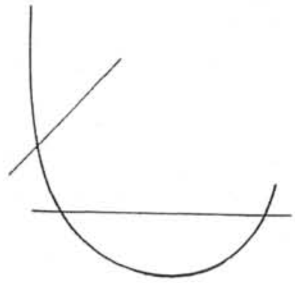


Fig. 5 Number of intersect.

b. Probability that a straight line intersects a line segment

(see fig. 8) The crossing number $n=1$ for all straight line in equation (9)

$$p_1 = 2\,l/L \tag{10}$$

p_1 is the probability that straight line intersects a line segment.

c. The measure of straight lines which intersect a convex curve.

If c is a convex, we have intersect number $n=2$ in equation (9) for all straight lines and the straight line which is at the position of contact c has zero measure.

$$\int dp d\theta = L \tag{11}$$

From the result of (**a**) and (**c**) in the above, one would obtain the next result;

d. If c_1 and c_2 are both convex curve and c_2 is contained in c_1, the probability that the straight line which intersect c_1 acrosses also c_2 is given by (see fig. 6)

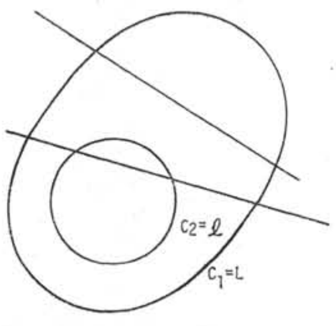


Fig. 6 The probability that random line intersects C_2 is given l/L.

$$p_2 = l/L\ . \tag{12}$$

p_2 ; the probability that the straight line intersects c_1 intersects also c_2

l ; the length of c_2

L ; the length of c_1

e. The probability that two straight lines intersect each other in convex figure c (see fig. 7)

Following the result of (7)

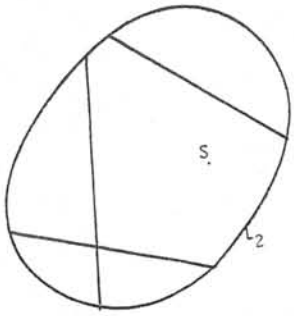


Fig. 7 The probbility that random line intersects each other is given $2\pi s/L^2$.

$$p_3 = 2\pi\, s/L \tag{13}$$

p_3; the probability that we want

s; areas of c

L; perimeter length of c

f. Example of application

When a road will be planned in a area whose perimeter is L, the probability that the road intersects the site of the facility (school or park etc.) whose perimeter is l will be given l/L under the condition that the road assigment is not determined but the necessity of the construction is decided. We must pay attention to the fact that the probability is given with no relation to the position of the facility and the road assignment. The probability is given with relation to their perimeter and has no relation to their area measurement.

3-3 Distribution of crossing number

In order to understand clearly the nature of the density of crossing nodes of a road or the number of bridges in a unit length along the river, we apply the results of the following.

a. Distribution in a unit length

In a domain c_1 of the perimeter L, which contains c_2 of the length l, the number of crossings that the straight line intersect c_2 is a poisson distribution. When n straight lines intersect c_1, the probability that k of n also intersect c_2 is given as follows; (fig. 8)

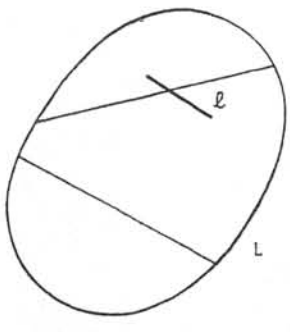


Fig. 8 The probability that random line intersects a segment is given $2l/L$.

$${}_nC_k P_1^k (1-P_1)^{n-k}$$

p_1; from the (12). $(p_1 = l/L)$

and so, $n \to \infty$

$$\frac{m^k}{k!} e^{-m} .$$

b. Distribution in a domain

In a convex domain of the perimeter L and area s, the probability that any two straight lines intersect each other in the domain is given by

$$p = 2\pi s/L^2 . \qquad \text{(see (13))}$$

Crossing number N that are realized in the convex by n straight lines is obtained from the following probability.

$$p(N=0) = (1-p)$$

$$p(N=1) = {}_N c_1 p^1 (1-p)^{N-1}$$

$$\vdots$$

$$p(N=k) = {}_N c_k p^k (1-p)^{N-k}$$

$$(N = {}_n c_2 = n(n-1)/2)$$

Note that crossing number N is binominal distribution.

When $n \to \infty$, $N \to \infty$, this equation becomes

$$N \sim \frac{m}{r!} e^{-m} \tag{15}$$

From the results above, the road-crossing number is Poisson distribution in a length or in an area, and that the bridge number along the river or railroad-crossing along the railway is Poisson distribution.

3-4 Distribution of length

Our next purpose is now to understand that the nature of intervals of crossing of

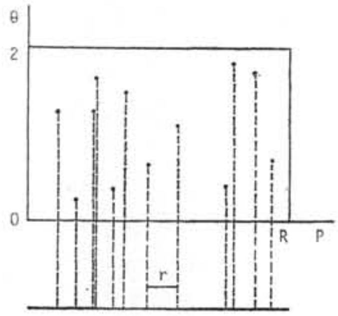


Fig. 9 Interval of random point to the random line.

road network or the accessibility to road or railway networks. Having obtained these, we shall study L-mosaic map.

a. Crossing interval

From equation (14), the crossing number is distributed according to Poissom distribution. Thus the interval of crossing x is given by

$$x \sim \rho e^{-\rho x} \tag{16}$$

where ρ ; the density of crossing number

b. Perimeter of the cell

When the crossing interval is an exponential distribution, the perimeter or k-angles convex is the sum of k exponential distributions. Therefore

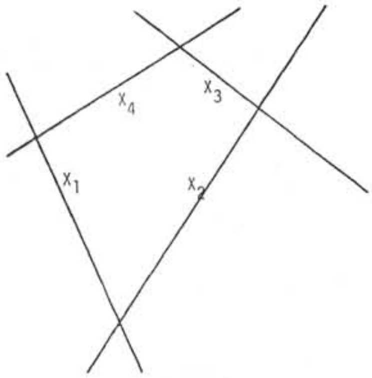


Fig. 10 Perimeter of the cell

$$x_i \sim \rho e^{-\rho x_i} \ (i=1, \cdots k)$$

$$y_k = x_1 + x_2 + \cdots\cdots + x_k$$

$$y_k \sim \frac{(\rho y_k)^{k-1}}{(k-1)!} e^{-\rho y_k}$$

so that, $y_k \sim \Gamma(k)$ distribution.

The perimeter of a cell will be regarded as the perimeter of an urban district.

c. Accessibility

We shall now consider that the nearest distance to random straight line from any point. According to the definition of random line, measure of the set of random lines holds constant for affin transformation. The density function of random lines is given by

$$f(p, \theta) \sim dpd\theta$$

where p is the distance from origin to the line. Hence accessibility r from any point to the nearest line is also an exponential distribution like the interval case.

$$r \sim \rho e^{-\rho r} \tag{18}$$

where ρ ; dencity of line

d. Diameter of a cell

In L-mosaic map, when one straight line is distributed randomly, the number of the crossing that the straight line intersect the mosaic map along the line is distributed by Poisson. Thus the interval is also exponential distribution. Moreover the diameter of the cell at any angle is also an exponential distribution.

3-5 The angle of crossing line

The angle ϕ of crossing lines is given by

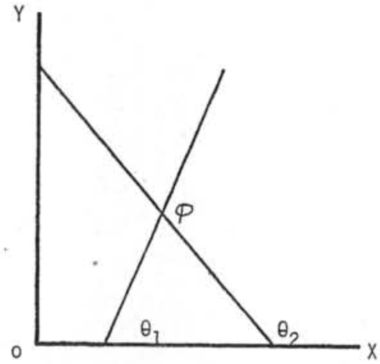


Fig. 11 Crossing angle of random line is ϕ.

$$f(\phi)=\frac{1}{2}\sin(\phi) \qquad (0\leq\phi\leq\Pi) \tag{19}$$

($f(\phi)=|1/2-\phi|$ *is not correct*)

(See referece, Santalo, Kurita.)

3-6 An area of cell

Since the distribution of a cell area has not been derived, we shall obtain the mean value and the second moment. The result is given by Goudsmit (1945)[9], but we derive the same result by an alternative method. We know Crofton's second theorem gives a clue to this problem[2].

a. Mean value

Mean value of an area of a cell is derived by the number of cells. A mean number of the crossing with n random lines in a convex is given from (15) by

$$N\cdot P=\frac{n(n-1)}{2}\cdot\frac{2\pi S}{L^2}$$

where S; area, L; perimeter, N; crossing number,

P; probability of intersect

The mean area of the cell is given by

$$\frac{S}{N\cdot P}=\frac{S}{\frac{n(n-1)}{2}\frac{2\pi S}{L^2}}=\frac{1}{\Pi}\cdot\frac{L^2}{n(n-1)}=\frac{1}{\Pi}\cdot\frac{L}{n}\cdot\frac{L}{n-1}$$

where L/n is transformed by the density of the line into $2/\rho$.

Therefore, we obtain,

$$E(s)=\frac{1}{\Pi}\cdot\frac{4}{\rho^2} \tag{20}$$

$E(s)$; mean value of cell area

b. Second moment

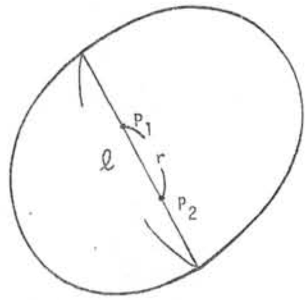


Fig. 12 Chord length and point distance r.

Crofton's second theorem (reference (2), (7)) proves that

$$\iint r^k dp_1dp_2=\frac{2}{(k+2)(k+3)}\cdot\int l^{k+3}dpd\theta$$

The left term implies that r is the distance between random two points p_1, p_2 in convex figure, and p_1 and p_2 can move at random all over the domain, dp_1dp_2. The right hand term implies that l is the length of any diameter and integral the diameter can move all over the domain, $dpd\theta$.

Now let $k=2$,

$$\iint dp_1dp_2=\frac{1}{3}\int l^3dpd\theta \quad , \quad \iint dp_1dp_2=s^2 \ .$$

When the number of the cell on L-mosaic map is N, the mean value of S is given by

$$\frac{E(s^2)}{3}=\sum_{i=1}^{n}\int l_i^3dpd\theta\Big/\sum_{i=1}^{n}\int dpd\theta$$

$$=\frac{1}{n}\sum_{i=1}^{n}s_j^2\Big/\frac{1}{n}\sum_{i=1}^{n}L_i$$

$$=L(s^2)/E(L)$$

where s ; the area of cell $\quad$ L ; perimeter of cell

$$E(s^2)=E(l^3)\times E(L)/3$$

From **3-4-d** and mean number angles of a cell is 4,

$$E(l^3)=\int_0^{\infty}l^3f(l)dl=\int_0^{\infty}l^3\rho e^{-\rho l}dl=3!/\rho^3$$

$$E(L)=4\times E(l)=4\rho$$

$$E(s^2)=8/\rho^4$$

c. Distance between two cells

The subject of the distance was declared on 3-4, but but by using Crofton's second theorem and Richard's idea (See reference (8)), we know another result.

First, let us define the k-adjacent. On any two cells in a mosaic map, when all line segments from one cell to the other intersect k lines, the two cells are designated.

We find the mean distance r between the k-adjacent cells.

$$E\left(\iint r_k^n dp_1 dp_2\right)=\frac{8}{\rho^{n+4}}\frac{(n+k+2)!}{k!}$$

where p_1 is in the cell and p_2 is in k-adjacent cell and the integral over the two point move in each cell. For $n=0$, $k=0$,

$$E\left(\iint dp_1 dp_2\right)=E(S^2)=8/\rho^4$$

$$E(r_k^n)=\frac{\Pi^2}{2}\cdot\frac{1}{\rho^{n+2}}\cdot\frac{(n+k+2)!}{k!}$$

When distance registance function of traffic generation is e^{-r}, summantion of traffic between k adjacent cells is given by

$$E\left(\iint e^{-r} dp_1 dp_2\right)=\frac{8(k+1)\rho^{k-2}}{(\rho+1)^{k+2}}$$

3-7 The relation of the morphological index

In the previous section, we introduced some morphological index of L-mosaic map. These are, however individually studied. In order to understand the network proportion or modular coordination, we shall hence combine the indexes and give a simple relation between them. For example, in crystals, Fullman gives the relation of the random division network as follow;

$$\frac{S}{V}=\frac{4l}{\Pi A}=\frac{2N}{L}$$

where N; nodes number, S; surface area
V; volume, L; edge length in a volume

From the right terms, we obtain

$$L=\sqrt{N\Pi S} \tag{25}$$

This formula means that the length is given by nodes and area. This can also be obtained from equation (20). L is length in area. $E(s)$ is mean area of a cell. $E(N)$ is mean number of nodes in area s. $E(l)$ is mean length of a edge.

$$E(s)=\frac{S}{E(N)}=\frac{4}{\Pi}\cdot\frac{L^2}{n(n-1)}=\frac{4}{\Pi}(E(l))^2$$

$$\bar{L}=2E(N)\cdot E(l)=2E(N)\sqrt{\pi\cdot S/4E(N)}=\sqrt{\Pi\cdot E(N)\cdot E(s)}$$

This implies that

$$\frac{(District\ area)}{(Length\ of\ road)}\propto\frac{(Length\ of\ road)}{(Crossing\ number)},\ \frac{S}{L}\propto\frac{N}{l}$$

This formula is the same form as the estimation formula of the length of minimum trees or minimum circuit in operations research.

$$L=c\sqrt{ns}$$

C; constant and for minimum trees, c is to 0.68, for minimum circuit, c is to 0.8.

4. Survey of Japanese Cities and Urban Module

4-1 Servey

Japanese famous 12 cities are chosen and from these urban maps—scale 1/25,000 —river, railway and road are abstracted (fig. 13-24). Number of nodes, length and

Fig. 13 Sapporo

Fig. 14 Sendai

Fig. 15 Yokohama

Fig. 16 Shizuoka

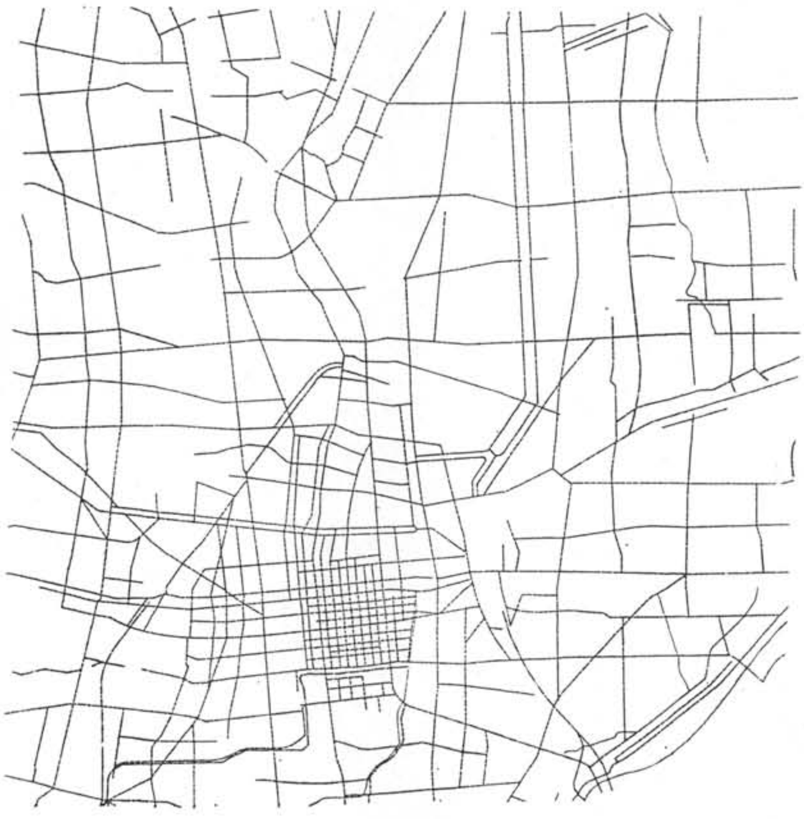


Fig. 17 Nagoya

Fig. 18 Kyoto

Fig. 19 Osaka

Fig. 20 Kobe

Fig. 21 Hiroshima

Fig. 22 Fukuoka

Fig. 23 Kumamoto

Fig. 24 Kagoshima

area are measured on the mean value and variance and distribution of each city. To measure the length, we use a planemeter, and read its value 1/10 of 10 mm unit. To measure the area, we put on 1 mm mesh and read the number of node in one unit.

4-2 Result of survey

a. Length

The mean value of link length is generally 670 meters on 12 cities and between them. The mean value of length is from 400 meters to 900 meters and its variance is a little different. On L-mosaic map, the distribution of link length is exponent the distribution. For scale effects on the difference and large scale road network is exponent but small road network is Γ distribution tendency.

b. Area

One urban district has about 30 ha on the average. The variance of area, we use $E(S^2)/(E(S))^2$ and estimate the variance. On L-mosaic Map, $E(s^2)/(E(s))^2=\pi^2/2=4.9$. Most cities have $3.5\sim5$.

Concerning the area distribution type, 12 cities have the the same tendencies.

But on L-mosaic map, the area distribution has not been theoretically.

c. Node number

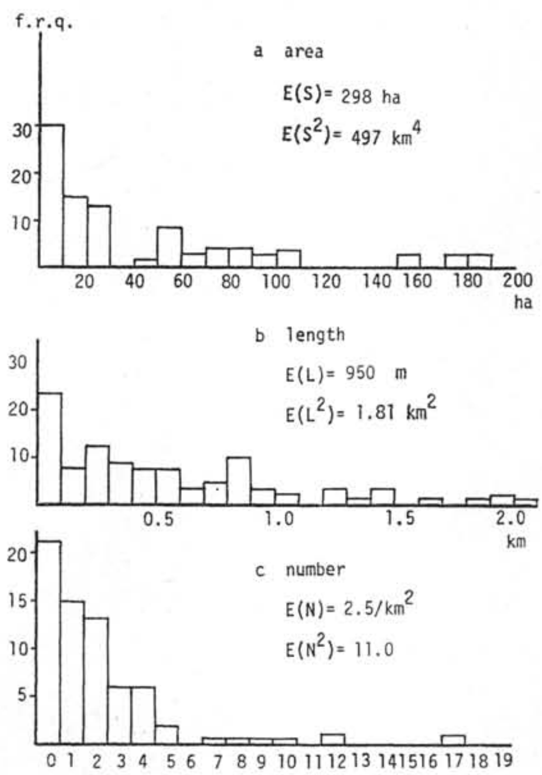


Fig. 25 Sapporo

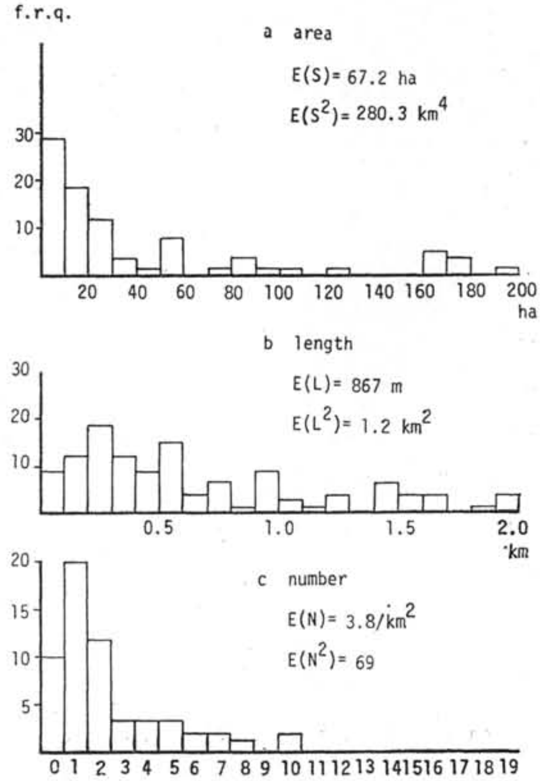


Fig. 26 Sendai

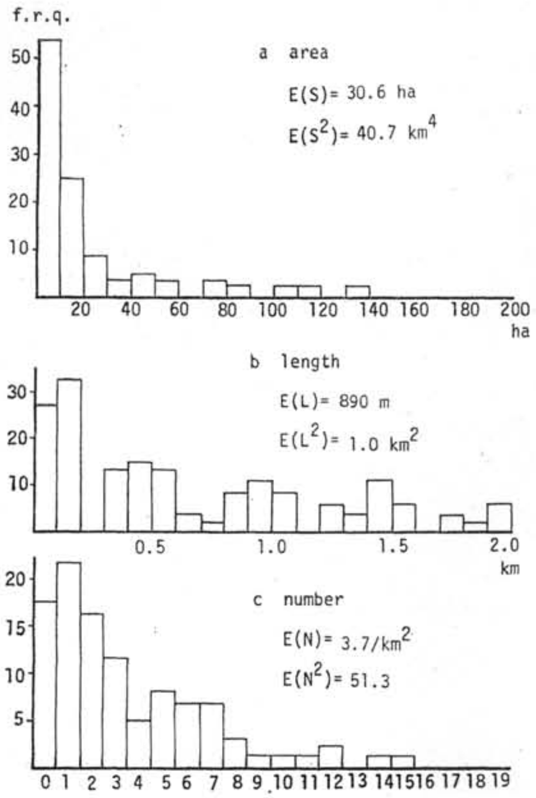


Fig. 27 Yokohama

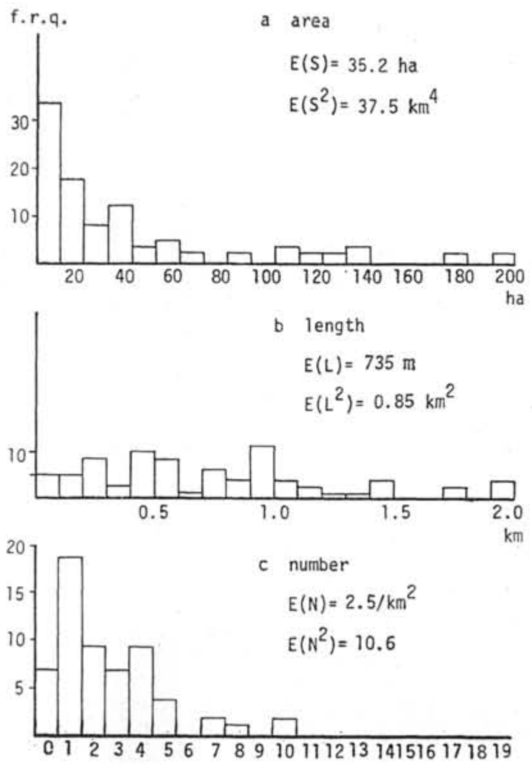


Fig. 28 Shizuoka

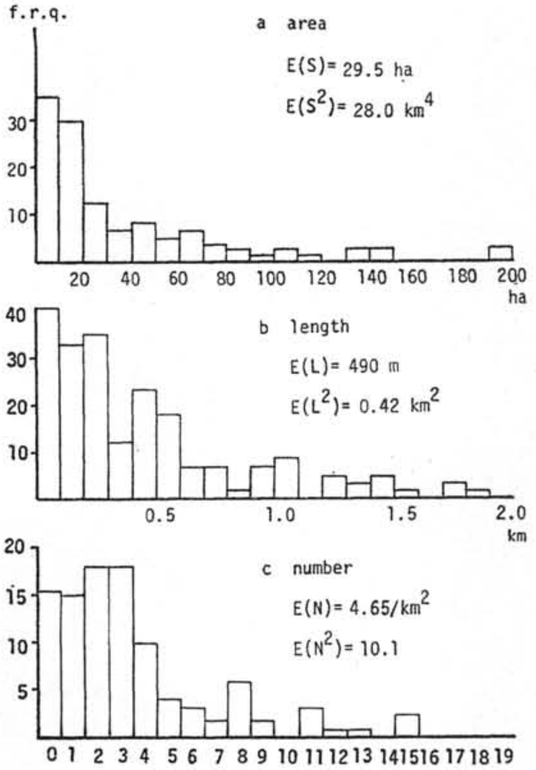


Fig. 29 Nagoya

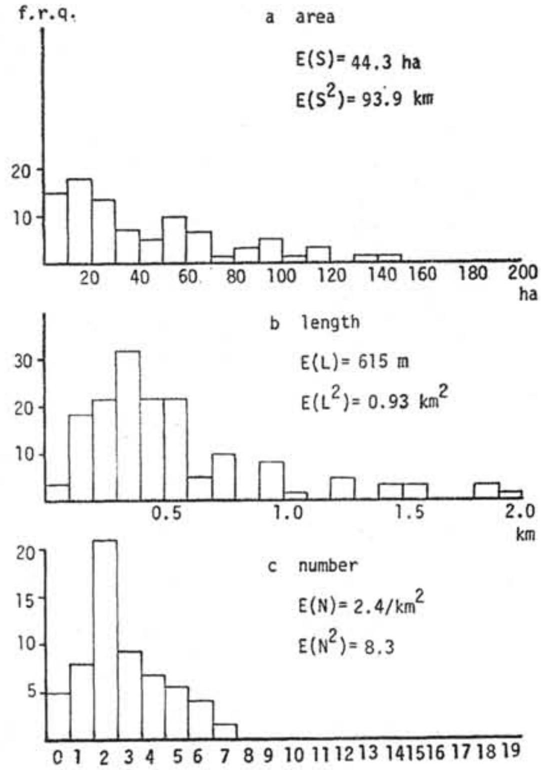


Fig. 30 Kyoto

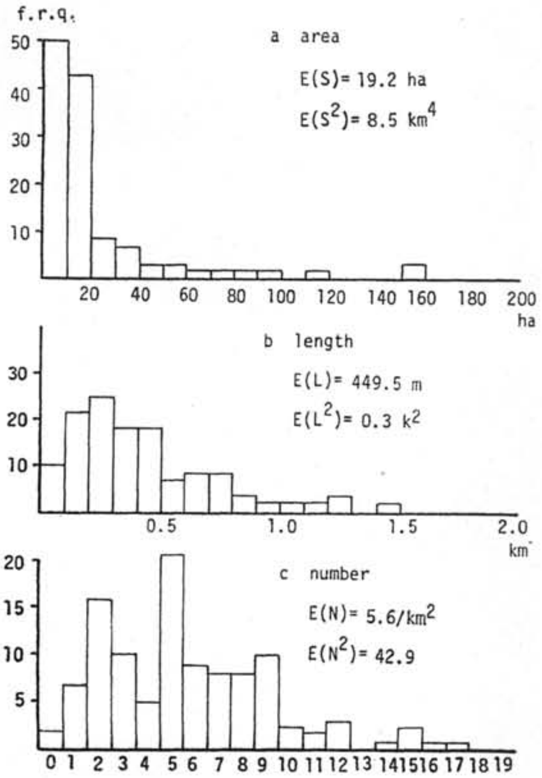


Fig. 31 Osaka

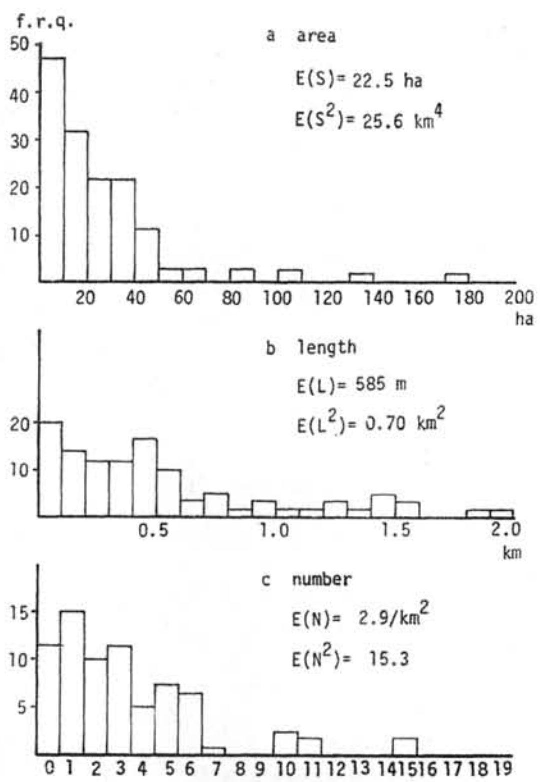


Fig. 32 Kobe

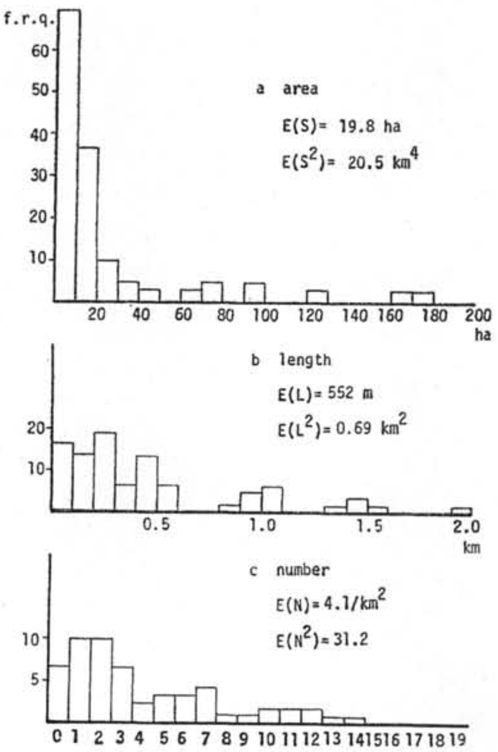


Fig. 33 Hiroshima

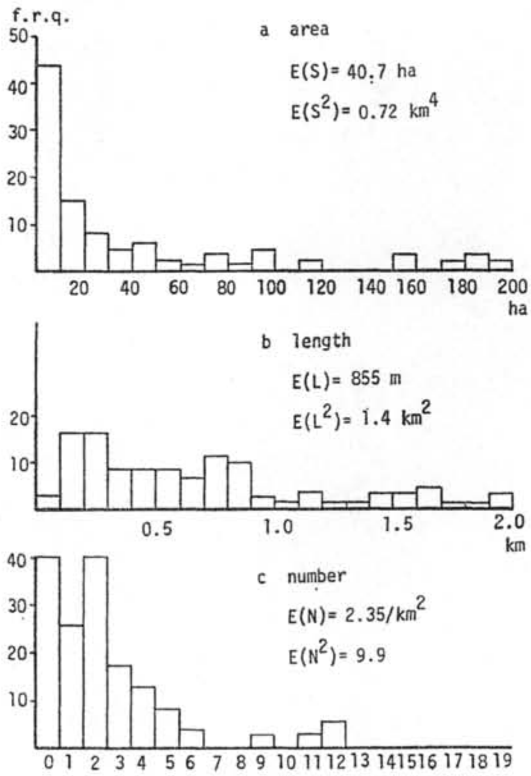
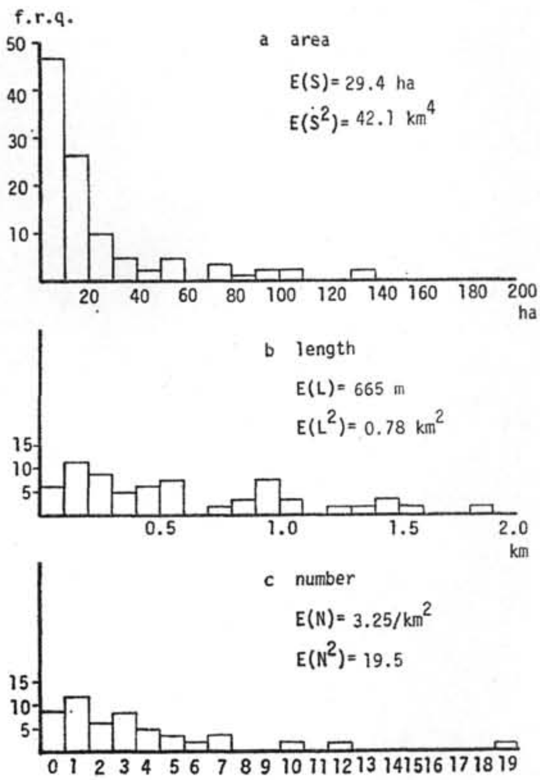


Fig. 34 Fukuoka



Fg. 35 Kumamoto

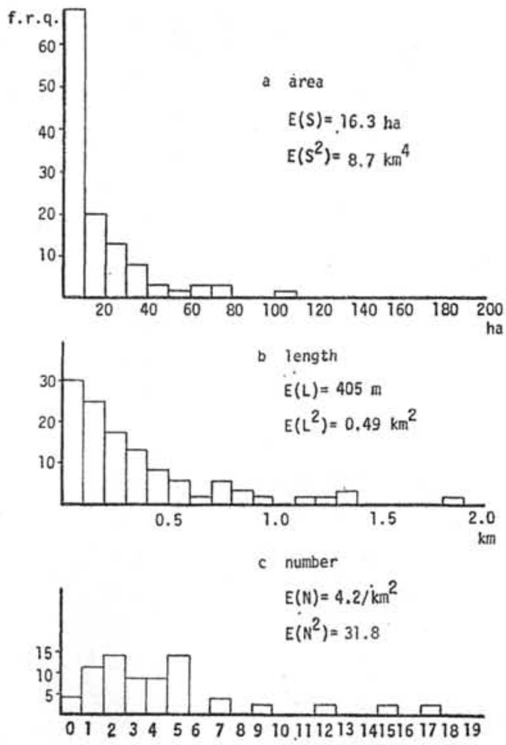


Fig. 36 Kagoshima

From L-mosaic model, we expect that this distribution will be poission type but the result shows Neyman-A distribution.

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Department of Housing and Environmental Design,
Faculty of Industrial Arts,
Kyoto Technical University,
Matsugasaki, Sakyo-ku, Kyoto

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