Lambert W Function Approach to Stability and Stabilization Problems for Linear Time-Delay Systems

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Contents

Chapter 1

Introduction

In the modern control theory, ordinary differential equations are usually employed to describe system dynamics. While they are well suited to many practical control problems, dymanical systems instinsically have some *delays* due to transmissions of signals, materials or informations. Most simply, if one uses feedback control techniques, the feedback structures naturally involve propagation delays. Differential equations that include time-delay factors are called *delay differential equations* or *time-delay systems* in terms of the control engineering. In practical situations, in addition to such feedback controllers, sensers, actuators or digital computers might be equipped in the system construction and these components also inhenrently have some time-delays owing to completions of their responses. Thus, the time-delays are essential features for system dynamics and therefore time-delay systems have been greatly concerned by many researchers and developed stability or robust stability analysis methods and control techniques up to now.

Unlike delay-free systems, behavior of time-delay systems depends on not only the present states but also the past ones. If one considers linear time-delay systems, their paticular characteristics are exposed by applying the Laplace transform. More specifically, they form certain transcendental functions in the frequency domain and have *infinitely-many characteristic roots*. Such infinitely-many roots are not perfectly computable in the practical viewpoint, so that representative ones have to be picked out for accounting their behavioral characteristics. Therefore, the state transitions are not completely transparent and thus effective methods are desired to expose the stability features. The above mentioned facts exhibit the distinctive difficulty in dealing with time-delay systems and give a motivation for the investigation.

Most of the existent stability analysis methods rely on the well-knowm Lyapunov's second methods, whose analogies in the context of time-delay systems are called Lyapunov-Krasovskii methods or Lyapunov-Razumikhin methods, in the time-domain. Alternatively in the frequency-domain, observation of the characteristic roots in the imaginary axis taking advantage of their parametric continuities. The former has a broad applicability regardless of whether linear or nonlinear and time-invariant or time-varying, whereas it suffered from conservativeness due to arbitrariness of the Lyapunov functions or functionals because the general forms can be hardly constructed. On the other hand, the latter often provides some complicated test algorithms or needs sweeping of the imaginary axis as the price for yielding exact results. As the same frequency-domain approach, the *Lambert W function*, which is a key tool of this thesis, has been first applied to linear time-delay system analysis earnestly, to the author's knowledge, in [4], although the suggestion has been made in [15] or more previously. The virtue of this function is to be able to explicitly express the characteristic roots of a class of linear time-delay systems, and make therefore exact and direct observation of them possible. Furthermore, each characteristic root can be easily computed with the help of Mathematica, Maple or Matlab Symbolic Math Toolbox, since they have a function for computing the Lambert W function. Due to these merits, the drawbacks of the above two approach can be overcome. However, this approach is suffered from a limitation of available system class. This point will be detailed later.

Turning to control problems, the Lyapunov methods are still available with the aforementioned advantages and disadvantages. As a pole placement method, finite spectrum assignment has been developed, which is categorized as the same schemes with Smith predictor or state predictor. Using these control methods, stabilization problems of linear time-delay systems are degenerated into delay-free ones, for which conventional pole placement techniques can be fully utilized. The Lambert W function offers a new pole placement technique such as reallocation of all the poles with preserving the infiniteness, i.e. without such a degeneralization to the delay-free case. A sort of the finite spectrum assignment for spectrally controllable systems requires time-integrations in the feedback configurations. However, in practical situations, integrations have to be numerically computed by quadrature methods, and it is reported in [96] that such quadratures cause destabilization of the closed-loop systems. This thesis proposes a new control scheme combinig the Lambert W function pole placement with non-predictive decoupling control of [87], which does not give rise to the above mentioned problem. This control scheme, furthermore, enables to compute stability delay margin of the closed-loop systems thanks to the uncanceled delay terms to which the softwares implementing the computation of the Lambert W function are applied.

In what follows, the organization of this thesis is in order.

In Chapter 2, firstly a short introduction of linear time-delay systems is given with their formulations and some applications of them are shown thereafter. In the subsequent sections, existent stability analysis methods and control techniques are overviewed and the standpoint of this thesis is also addressed. Then, a formal introduction of the Lambert

W function is given, stressing the important property of this function for the stability analysis. Finally, how to adopt the Lambert W function as a tool for stability analysis of linear time-delay systems is shown and fundamental stability and robust stability criteria are presented.

In Chapter 3, robust stability criteria are derived based on the fundamental criteria. The obtained criteria result in *extreme point results* and *boundary implications* in the presence of prescribed uncertainties. The extreme point result implies that stability all over uncertainty regions can be determined on only some boundary points in the regions. The boundary implication is a generalization of the extreme point result in such a way that stability all over prescribed uncertainty regions is determined on their boundaries. They both facilitate robust stability check, whereas it should be remarked that available systems for these stability criteria are rather restricted due to the definition of the Lambert W function.

In Chapter 4, a stabilization technique for linear time-delay systems is developed by means of the Lambert W function. For this purpose, *decoupling* control is applied as a controller design technique. The reason of using the decoupling control is that it can modify given linear time-delay systems into the suitable forms to the Lambert W function approach. In this sense, it can overcome the inherent rectriction of this function with respect to available system class. Based on the decoupling control, a new pole placement technique using this function is proposed for the aim of stabilization of the decoupled systems. In this context, the proposed stabilization method may well be said to be a combined method of the Lambert W function approach with the decoupling control.

In Chapter 5, as a possible application of the present approach, additional dynamics induced by some model transformations of linear time-delay systems is investigated by the Lambert W function. Such model transformations are introduced in order to derive delay-dependent stability conditions in the use of the Lyapunov methods. Additional dynamics arises when the model transformations are applied to the target time-delay systems and this causes conservativeness of the obtained stability conditions. Nevertheless, if the additional dynamics is stable, such stability regression does not occur. This fact motivates to study the stability of the additional dynamics. Although the additional dynamics stability has been well researched by some authors [24–26, 47, 48, 50], this thesis gives a new insight using the Lambert W function into the study. As shown there, this function is again well suited to the additional dynamics analysis and able to analyze the stability characteristics of the additional dynamics of several types of model transformations with several advantages. Especially the first-order and the second-order transformations are studied in this thesis.

The thesis is concluded in Chapter 6 with remarks and some comments concerning future developments.

Chapter 2

Background and Preliminaries

2.1 Linear Time-Delay Systems

In the control engineering field, a linear system

$$\dot{x}(t) = Ax(t) \tag{2.1}$$

is usually employed as a system model for estimating and improving system performance, where A is an appropriate dimensional matrix, $x(t)$ system states and t time of the system. However, most of artificial systems naturally include some *delays* due to transmitting signals or materials, estimating control signals, sensor responses and actuator motions. If one considers feedback controlled systems, time-delays exist in any components of the system configurations as shown in Figure 2.1. If such time-delays are sufficiently small, crucial troubles may not arise. However, if not, it is possible that time-delays not only deteriorate system performance but also even destroy the stability of system behavior [6, 24, 28]. This shows the importance of investigating time-delay systems. It is common to pick out dominant delays and discard uninfluential ones when modeling and controlling time-delay systems.

Taking time-delays into account, a *linear time-delay system* or *linear retarded time-delay system* is defined by

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \tag{2.2}$$

where A and B are coefficient matrices, $x(t)$ system states and $\tau > 0$ stands for a time-delay. If $\tau = 0$, the system (2.2) corresponds to the delay-free system (2.1) by replacing A of (2.1) by $A + B$ of (2.2).

If an initial state (or value) $x(0)$ is given to the system (2.1), there exists a unique solution with respect to the state variables. On the other hand, for the existence of a

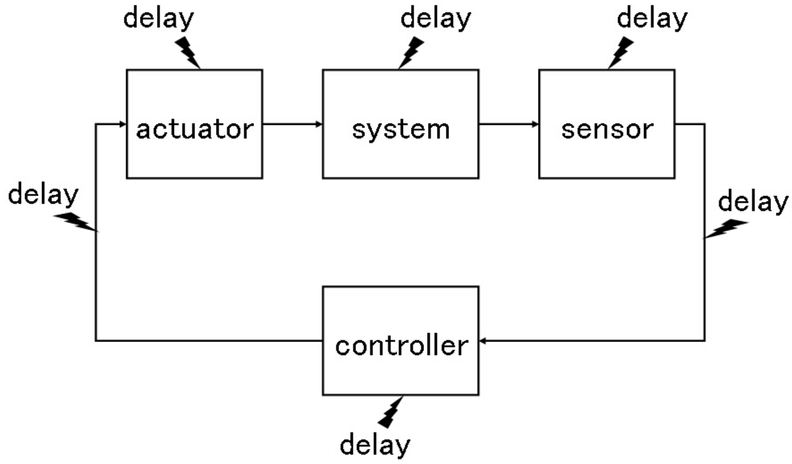

Figure 2.1: Actual feedback controlled systems.

unique solution in the system (2.2), an initial function, not a value, must be given in the interval from τ past to the initial time 0:

$$x(t) = \phi(t), \qquad t \in [-\tau, 0]. \tag{2.3}$$

In order to solve the differential equation (2.2), let us bring it to the frequency domain by the Laplace transform. Letting $\mathcal{L}[\cdot]$ be the Laplace transform, the system (2.2) can be described in the frequency domain as

$$sX(s) - \phi(0) = AX(s) + B\left(e^{-\tau s}X(s) + \int_0^\tau e^{-st}\phi(t-\tau)dt\right), \tag{2.4}$$

where $X(s) = \mathcal{L}[x(t)]$. Further modify (2.4) to

$$\begin{aligned} & (sI - A - Be^{-\tau s})X(s) = \phi(0) + B\int_0^\tau e^{-st}\phi(t-\tau)dt \\ \Leftrightarrow\ & X(s) = (sI - A - Be^{-\tau s})^{-1}\left(\phi(0) + B\int_0^\tau e^{-st}\phi(t-\tau)dt\right). \end{aligned} \tag{2.5}$$

Accordingly, the solutions can be obtained by operating the inverse Laplace transform to $X(s)$ in (2.5). For this, the characteristic equation

$$\det[\, sI - A - Be^{-\tau s}\,] = 0 \tag{2.6}$$

must be solved to find the poles for applying the residue theorem. It is well known that the equation (2.6) has infinitely many solutions and cannot be solved algebraically. Therefore, one has to rely on numerical methods, but the solutions are never exhibited in the complete forms in the practical sense.

Even though the solutions of (2.2) are not in hand, the stability can be examined with the characteristic equation (2.6). A definition of stability or pricisely asymptotic stability of the system (2.2) is as follows:

Definition 2.1 The linear time-delay system (2.2) is *stable* (or precisely *asymptotically stable*) if

$$\lim_{t \to \infty} x(t) = 0. \tag{2.7}$$

Relating to the characteristic equation, stability is redefined as in the following definition.

Definition 2.2 The linear time-delay system (2.2) is *stable* (or precisely *asymptotically stable*) if all the solutions of (2.6) lie in the complex open left half-plane $\mathbf{C}^-$. Furthermore, the real part of the rightmost solution is called *stability exponent*, that represents the effect of the most dominant characteristic root on the system behavior.

Some generalizations can be made by adding more delayed terms to (2.2) as

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - i\tau), \tag{2.8}$$

where A_i's are coefficient matrices. This system is called *commensurate linear time-delay system*. In this thesis, the stability of the system (2.8) is not analyzed because it is cannot be dealt with by the Lambert W function. However, stabilization of it is possible if some appropriate structural reconstruction can be made by feedback controllers. This subject is deferred to Chapter 4. If the states x of (2.8) are delayed by at least two base time-delays which are independent each other, such systems are called *incommensurate time-delay systems* which make the analysis more complicated.

By defining a delay operator, a convenient representation of the system (2.8) can be obtained. The delay operator Δ is defined as

$$\Delta x(t) = x(t - \tau). \tag{2.9}$$

Using the operator Δ, the system (2.8) is expressed as

$$\dot{x}(t) = \sum_{i=0}^{N} A_i \Delta^i x(t). \tag{2.10}$$

Letting

$$A(\Delta) = \sum_{i=0}^{N} A_i \Delta^i, \tag{2.11}$$

(2.10) is rewritten as an analogy to the delay-free system (2.1):

$$\dot{x}(t) = A(\Delta)x(t). \tag{2.12}$$

The Laplace transform of Δ is quite comprehensible. Operating $\mathcal{L}$ to (2.9) results in

$$\mathcal{L}[\,\Delta x(t)\,] = e^{-\tau s}X(s). \tag{2.13}$$

Now, setting $\sigma = e^{-\tau s}$, (2.13) is written as

$$\mathcal{L}[\,\Delta x(t)\,] = \sigma X(s). \tag{2.14}$$

Roughly speaking, σ is the Laplace transform of Δ. By virtue of (2.14), the characteristic equations of (2.2) and (2.8) can be expressed in terms of σ as

$$\det[\,sI - A - B\sigma\,] = 0 \tag{2.15}$$

and

$$\det\,[\,sI - A(\sigma)\,] = 0 \tag{2.16}$$

respectively.

2.2 Applications of Time-Delay Systems

Time-delay systems have great potential to practical applications as imagined from their natural features.

As the most basic applications of time-delay systems, process control in *chemical industries* should be mentioned [72, 106]. It usually has delays due to material transportations, thus time-delay systems are more suitable as system models than delay-free systems.

In the last two decades, the vast number of papers have been published on the subject of *networked control systems* [36, 59, 99, 101]. If feedback controllers were apart from the objective systems in the distance and connected with networked lines, managing the communication delays in the data transmissions became an issue and therefore it was the case of applying time-delay systems. In the networked systems, as the standard configurations, there are two independent delays being possibly time-varying or probabilistic within the feedback loop. Moreover, packet loss and bandwidth constrains should be also taken into account in the networked control systems.

Sometimes in the networked control systems, feedback control tasks might be left to a human operator. The human might require the haptic informations from the target systems so that he can operate as he wants. This control scheme is called *bilateral teleoperation* expected to utilize for missions in the space, deep-sea, dangerous zones and medical practices [33, 73, 92].

From the other aspect, controlling date transmissions may be targeted: this scheme is called *congestion control*. As a congestion control strategy, the TCP (Transmission Control Protocol) is usually used in the Internet. However, the TCP sometimes causes long and variable delays due to retransmissions of lost packets and therefore it is not suitable in real-time data transfer such as video streaming. For the purpose of compensating this insufficiency of the TCP, some researchers have made attempts to add feedback control structures to network configurations, resulting in feedback controlled time-delay systems [5, 64, 97].

Recent developments of the applications to the network controlled systems are well showcased in [14].

Furthermore, there are many engineering applications as enumerated in the following: internal combustion engines [24,43], active suspensions [38], neural networks [62,72], chatter control in metal cutting processes [24, 103], semiconductor lasers [60] and so on [28, 106]. Besides engineering, time-delay systems are frequently exploited for describing biological phenomena such as growth of a single species, spread of infections and circulatory systems of humans [28, 72, 82, 106].

As mentioned previously, time-delays may destabilize the systems. However, there are interesting utilizations of time-delay systems that may sound paradoxical; that is to use the time-delays for the purpose of stabilization.

Delayed feedback control has made an impact on the controlling chaos issue [81, 100]. This control method consisted of difference feedback between present states and one period past states on periodic orbits. While the delayed feedback control did not need precise knowledge of periodic orbits such that claimed in the OGY method [78], it suffered from the so-called odd number limitation. In [32, 54, 55], the delayed feedback control has been adapted to stabilization problems of uncertain steady states involved, for example, in inverted cart-pendulum systems on a varying slope or passive walking robots descending a gentle slope.

Another application was a sort of active vibration absorbers called *delayed resonator* [24, 76]. It consists of delayed feedback controller inserted into oscillatory systems for absorbing vibrations more efficiently.

From the theoretical viewpoint, it has been proven that a second-order oscillatory system $\ddot{y}(t) + \omega_0^2 y(t) = u(t)$, which could be stabilized with derivative feedback $u(t) = k\dot{y}(t)$ but never stabilized with proportional feedback $u(t) = ky(t)$, could be also stabilized with delayed feedback $u(t) = ky(t-\tau)$ where $\tau > 0$ is a time-delay [1, 72]. This result have been extended to higher-order differential equations in [52, 74].

2.3 Existent Stability Analysis Methods

In this section, existent stability analysis methods for time-delay systems are overviewed.

If one considers to apply the classical techniques for delay-free systems such as the Nyquist plot and the Routh-Hurwitz criterion for delay systems, the former would be available, indeed it is often used for stability performance check of the systems, but the latter is not practical [72]. Generally, to apply the delay-free techniques involves difficulties since time-delay systems are infinite dimensional. This fact motivates to develop simple and easily computable stability criteria for time-delay systems.

The *Lyapunov-Krasovskii* method is one of usual methods for stability analysis of time-delay systems [19,29,31,40,56,57,80]. The advantage of this method quite lies in its genaral applicability. It is almost free from a system class restriction; it can be applied to linear, nonlinear, time-invariant and time-varying systems, and provide *delay-independent* and/or *delay-dependent*[1] stability criteria according to system specifications. A recent tendency of this direction is to modify the existence conditions of the Lyapunov-Krasovskii functionals to some tractable *linear matrix inequality (LMI)* conditions. This tendency seems to be strengthened by the development of efficient algorithms for solving LMIs [7], which is actually implemented by the software such as Matlab or Scilab. However, this approach involves intrinsic conservativeness due to restrictions on forms of the Lyapunov-Krasovskii functionals for producing LMI conditions. This fact leads to a trade-off between computational accuracy and simplicity. In [49, 51], Kharitonov *et al.* have tackled constructing problems of the complete Lyapunov-Krasovskii functional [28], which provides necessary and sufficient stability conditions, whereas they faced difficulties in resolving the essential complexity of the complete construction. In the last decade, numerous papers have been published in order to reduce the conservativeness of the Lyapunov-LMI method. As a result, key points of the reducing techniques could be narrowed down to three [40, Introduction]. As an alternative to the Lyapunov-Krasovskii functional, the *Lyapunov-Razumikhin function*, not functoinal, is sometimes used to derive LMI stability conditions, nevertheless the Krasovskii type approach has the majority in the literature since the Razumikhin type generally yields conservative results than the Krasovskii one.

It should be noted that in [56, 57] *model transformation* techniques are adapted for obtaining delay-dependent and/or delay-independent stability conditions. However, the model transformation gives rise to consevativeness of the resultant conditions, because

[1] The term delay-independent means that stability or other conditions do not include time-delay amounts, i.e. conditions satisfied for any time-delays systems. Otherwise, they are called delay-dependent conditions. If one has no time-delay information, then the delay-independent types work efficiently. If not, one had better use delay-dependent conditions rather than the delay-independent ones because the latter generally leads to conservative results.

additional dynamics is brought into the transformed systems. If the additional dynamics is stable, then the stability of the transformed systems is equivalent to the original one. Although the stability of the additional dynamics is fully investigated in [24–26, 47, 48, 50], a new insight into this analysis technique is given in Chapter 5 by using the Lambert W function. This function yields an explicit expression of the eigenvalues of the additional dynamics.

Other than the Lyapunov methods, various types of stability analysis methods have been also investigated.

In [68, 69], easily checkable criteria have been derived by a comparison of the matrix measure and the matrix norm. The *Tsypkin's* criterion [72] was also a basic result for delay-independent conditions of rational transfer functions with an input delay.

In [41, 42], Kamen has established fundamental delay-independent stability criteria for commensurate time-delay systems. He used the *two-variable quasi-polynomials* generated by replacing the exponential factors in the characteristic functions by an individual variable. Chen et al. have improved the Kamen's results by returning to the original quasi-polynomials from the two-variable one, and constructed computationally effective methods, so-called *frequency-sweeping tests* [10–13]. By carrying out the tests, one could check delay-independent and/or delay-dependent stability of either commensurate or incommensurate systems and even robust stability.

Olgac's group has evolved computational strategies for estimating complete stability delay margin based on the Rekasius' lemma [18, 77, 84]. In paticular, in [18] they introduced a notion of *building block* such that it could separate infinitely extended time-delay spaces where the characteristic functions were vanished on the imaginary axis into a finite kernel space and its copies by utilizing the periodicity of exponential terms with a pure imaginary number, and organized a graphical test using this notion in which one could observe a stable map spanned by incommensurate time-delay axes.

Now let us focus on robust stability. Robust stability of time-delay systems usually has two meanings: one is time-delay robustness and the other is that of system coefficients. Conventionally, if one says "time-delay systems are robustly stable", then it usually imply robustness of system coefficients and time-delay robsutness is regarded as delay-dependent stability. In what follows, the review proceeds in this manner.

For robust stability analysis, the Lyapunov-Krasovskii and Lyapunov-Razumikhin methods are still available. Indeed, robust stability LMI conditions can be obtained by a similar way to the case of the nominal stability [19, 29, 31, 40].

In eigenvalue analysis viewpoints, the *Edge theorem* can be proven in terms of linear time-delay systems [20]. The Edge theorem is such that if quasi-polynomial families are configured polytopically, the families are stable if and only if their every edge quasi-polynomials are stable. The subsequent paper [44] has concerned with *convex directions* of segment quasi-polynominals under which a segment quasi-polynomial is stable if and only if its vertex quasi-polynomials are both stable. The number of test quasi-polynomials justifying the Edge theorem can be reduced by means of the convex directions. If given polytopes of quasi-polynomials can be specified as interval or diamond forms, some further reductions or so-called "implications" of stability can be made [44, 45, 53].

The real or complex *stability radii* measure allowable perturbations for preserving the stability of linear time-delay systems [35, 71]. Although the complex stability radii are unpractical and conservative, their computations can be implemented by more tractable tasks than the real ones in general.

For further results, refer to the textbook [24], the survey papers [27, 46, 72, 85] and references therein.

Finally, let us introduce a new tool in this research field named *Lambert W function* (or *product log function* in terms of the computer software Mathematica) which has received some researchers' attention among not only time-delay system researches but also the other engineering fields and mathematical problems since it was given the name in [15]. To the author's knowledge, [4] was the first paper that exploited the function for stability analysis of time-delay systems in earnest. This function is a key tool throughout this thesis too. More details of this function are addressed in Section 2.5.

2.4 Existent Stabilization Methods

Let us review existent stabilization methods for time-delay systems in this section.

Even for time-delay systems, classical *PID control* is still workable in a large number of practical situations, especially industrial plants [2, 106].

Smith predictor is also a classical but fundamental tool for controlling time-delay systems [2, 106]. This is a unique tool for time-delay systems, because this estimates predicted outputs against time-delays. Resultant systems after the predictions behave as if they have no time-delay; as a result one can take advantage of design methods for delay-free systems. However, the Smith predictor based control restricts objective systems to only stable ones. *Modified Smith predictor* removes this restriction, i.e. it can be applied to even unstable systems with certain (troublesome) approximations [106].

State predictor is an analogy to the modified Smith predictor for state-space models. It estimates not future outputs but future states of the target systems [22, 75]. As expected, resultant closed-loop systems match to delay-free systems. Theoretically speaking, state predictor does not claim that stable systems are preassigned. More precisely, if the feedback law is completely computed, the closed-loop systems can be stabilized whether the target systems are stable or not; however that is unrealistic in the implementation stage as suggested below.

Finite spectrum assignment is such pole placement design method that infinitely destributed pole maps of linear time-delay systems are rearranged into finite ones; resultant systems are like delay-free. If given systems are *reachable*, then finite spectrum assginment can be accomplished by a procedure of [70] (for Bezout approach [16]). Finite spectrum assignment for *spectrally controllable* systems, which include reachable systems, has been thoroughly studied too [63, 98] (for Bezout approach [93]). If systems are only *stabilizable* that is a relaxed condition of reachability, then a similar design can be made in the price of leaving infinitely many stable uncontrolled poles [21].

In the implementation stages of the modified Smith predictor, state predictor and finite spectrum assignment for spectrally controllable systems, integration operations are demanded in the feedback loops so that desired input signals can be adequately provided. In practical situations, however, integrals must be calculated by some numerical method. If quadrature methods such as rectangular, trapezoidal or Simpson methods have been employed for the calculations, it has been comfirmed in [96] that such the numerical approximations might break down the stability of the closed-loop systems. This phenomenon has been well analyzed in [17, 65, 66] and the textbook [106] and they have also developed some techniques to overcome this shortcoming. Note that the procedures for reachable or stabilizable time-delay systems (the Morse's procedure [70] etc.) do not cause the above mentioned problems since the designed controllers would not include any integral terms.

H_∞ control scheme has been established for time-delay systems as well as delay-free systems. The H_∞ control problems could be formulated into solvablity conditions of algebraic Reccati equations constrained via design parameters [58, 106].

The *Lyapunov-Krasovskii* (and the *Lyapunov-Razumikhin*) methods are available again for controller synthesis [67, 105] and even H_∞ problems [19, 61]. As an analogy to the stability analysis, LMI conditions for stabilization and/or robust stabilization can be derived by a common way.

To the best of the author's knowledge, control methods using the Lambert W function have not been well studied. For this reason, in Chapter 4, this thesis presents a Lambert W function method that combines a new pole placement technique using this function with a decoupling control scheme of [87].

2.5 Lambert W Function Approach

The Lambert W function has been familier in the field of engineering, physics and mathematics since it was given the name in [15] or the caluculation module was built in the computer algebra software Maple in some years before. For instance, it has been shown that the Lambert W function could describe a solution to a problem of computing the range of projectile in viscous fluids [79]. Besides, it can be well suited to linear time-delay system analysis as desired in this thesis; for this direction, see the literature [4,37,39,102,103]. For the other applications, see [8,9,15,23,30,86,95] and references therein.

The merit of using the Lambert W function quite lies on that it can algebraically solve the characteristic equations of scalar linear time-delay systems [4]. This makes one possible to explicitly express the characteristic roots of the systems and easily compute them with the help of Mathematica, Maple or Matlab which has a function to calculate the Lambert W function. Moreover, it helps to study the qualitative features of them and always supplies an exact analysis being free of conservativeness. The aforementioned stability analysis methods almost rely on certain simplifications causing conservativeness (for example the Lyapunov-Krasovskii method), some complicated algorithms for providing exact solutions or observation of the characteristic roots on the imaginary axis neglecting the distance from the imaginary axis. It is clear that the Lambert W function approach has an advantage over those methods. However, it imposes a restriction to a class of available systems and this point is the main drawback of the Lambert W function approach.

In this thesis, to overcome the restriction on the available system class, *decoupling* control is utilized for the purpose of making changes of system structures. In this strategy, stablization is simultaneously achieved with decoupling by making use of the Lambert W function. This method can be compared with the predictive control such as the finite spectrum assignment. While the finite spectrum assignment based on spectrally controllable systems requires integral terms in the feedback laws, the Lambert W function approach combined with the decoupling control is free of them, as a result, it does not give rise to the implementation problems. Moreover, by leaving the delay effect in the closed-loop configurations, it enables to be easy to compute delay margin for stability of the closed-loop systems. It should be emphasized that the proposed pole assignment can be done as easier as the finite spectrum assignment thanks to a certain crucial property of the Lambert W function.

The next subsection explains about the Lambert W function and its crucial property for stability analysis of linear time-delay systems. In the subsequent subsection, underlying theories on the stability analysis using this function are addressed.

2.5.1 Lambert W Function

This subsection is proceeded based on the introduction paper of the Lambert W function [15].

The *Lambert W function* (or *product log function*) is defined as the solutions $w \in \mathbf{C}$ of the equation

$$we^w = z \tag{2.17}$$

for $z \in \mathbf{C}$ and denoted by the symbol W, i.e.

$$w = W(z). \tag{2.18}$$

As indicated by the name "product log function", it is an extension of the logarithm. The numerical computation procedure attaining a sufficiently precise evaluation has been established in Mathematica, Maple and Matlab Symbolic Math Toolbox.

The equation (2.17) has infinitely many solutions, namely the Lambert W function is an infinitely many-valued function, in other words it has an infinite number of *branches*. Each branch is distinguished by a subscript k as W_k, $k = 0, \pm 1, \pm 2, \cdots, \pm\infty$, especially W_0 is called *principal branch*. For the sake of simplicity, W_∞ and $W_{-\infty}$ are regarded as "a function" respectively. In fact, $\lim_{k\to\pm\infty} W_k(z)$ tends to complex infinity and the above simplification does not lead to inconsistency. The range of each branch is shown in Figure 2.2.

Remark 2.1 The range of the Lambert W function is symmetric with respect to the real axis.

The boundaries depicted in Figure 2.2 are images of *branch cuts* in the z-plane. Let us define the branch cut of each branch in the following. W_0 has a branch cut linking to W_1 and W_{-1} which is defined as

$$B_{C0} := \left\{ a + j0 \;\middle|\; -\infty < a \le -\frac{1}{e} \right\}. \tag{2.19}$$

As seen from Figure 2.2, W_1 adjoins to W_0 and W_{-1} in the lower side and to W_2 in the upper side and W_1 has three branch cuts B_{C0} as a link to W_0,

$$B_{C1} := \left\{ a + j0 \;\middle|\; -\frac{1}{e} < a \le 0 \right\} \tag{2.20}$$

to W_{-1} and

$$B_C := B_{C0} \cup B_{C1} \tag{2.21}$$

to W_2. W_{-1} has the same but upside-down branch cuts to W_1. The other branches W_k, $k = \pm 2, \pm 3, \cdots, \pm\infty$ have the branch cut B_C in both of the lower and upper sides. In Figure 2.3, 2.4 and 2.5, the correspondences between the z-plane and the w-plane in terms of W_0, W_1 and W_{-1} are shown respectively. Restricting the argument in the z-plane to $(-\pi, \pi]$, the image of the argument $+\pi$ of the branch cut represented by the bold line in

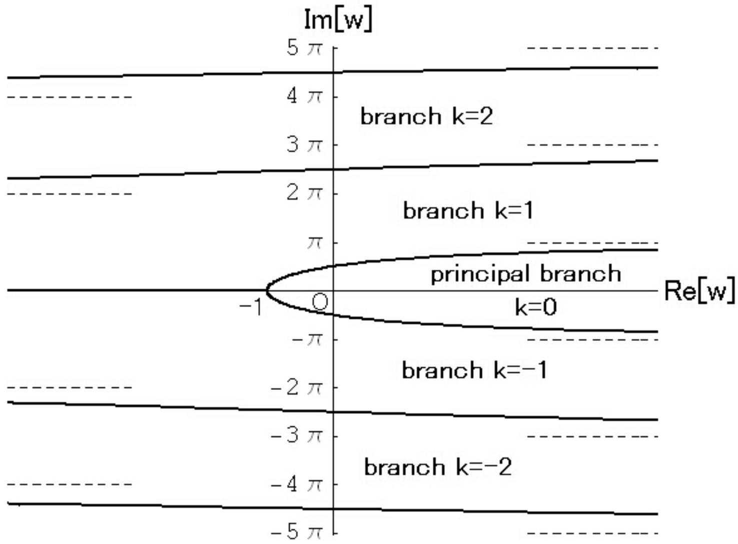

Figure 2.2: Ranges of W_k, $k = 0, \pm 1, \pm 2, \cdots$.

the z-plane corresponds to the upper boundary of W_0 in the w-plane and the argument $-\pi$ represented by the dashed line corresponds to the lower boundary in these figures. The markings A-F indicate the corresponding points between the two planes. The ranges of the quadrants in the z-plane are also written in the w-plane.

Remark 2.2 When a curve, say L, which has no intersection to the branch cuts with the z-plane is mapped by W, the images $W_k(L)$, $k = 0, \pm 1, \cdots, \pm\infty$ describe continuous curves and W_k, $k = 0, \pm 1, \cdots, \pm\infty$ are bijective.

The following lemma plays a key role throughout the thesis.

Lemma 2.3 For $z \notin B_{C0}$,

$$\max_{k=0,\pm 1,\cdots,\pm\infty} \mathrm{Re}[\,W_k(z)\,] = \mathrm{Re}[\,W_0(z)\,] \tag{2.22}$$

holds. For $z \in B_{C0}$,

$$\max_{k=0,\pm 1,\cdots,\pm\infty} \mathrm{Re}[\,W_k(z)\,] = \mathrm{Re}[\,W_0(z)\,] = \mathrm{Re}[\,W_{-1}(z)\,] \tag{2.23}$$

holds.

Lemma 2.3 can be observed from Figure 2.6 intuitively. Figure 2.6 shows curves $W_k(C_r)$, $k = 0, \pm 1, \pm 2, \cdots$ where

$$C_r := \{re^{j\theta} \mid \theta \in (-\pi, \pi]\} \tag{2.24}$$

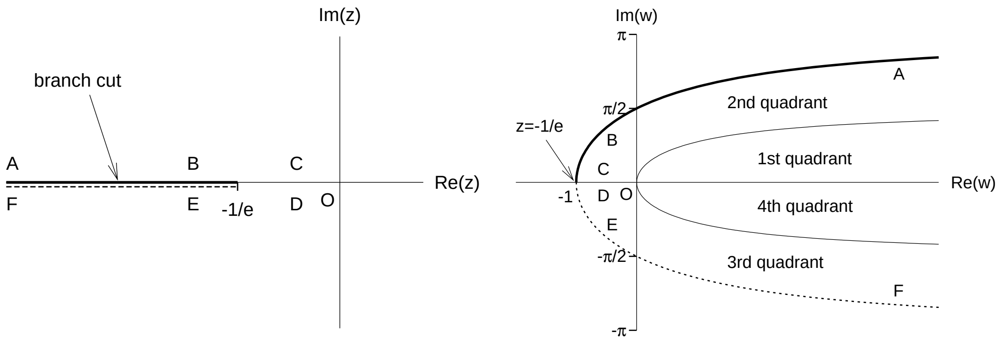

Figure 2.3: Mapping by W_0.

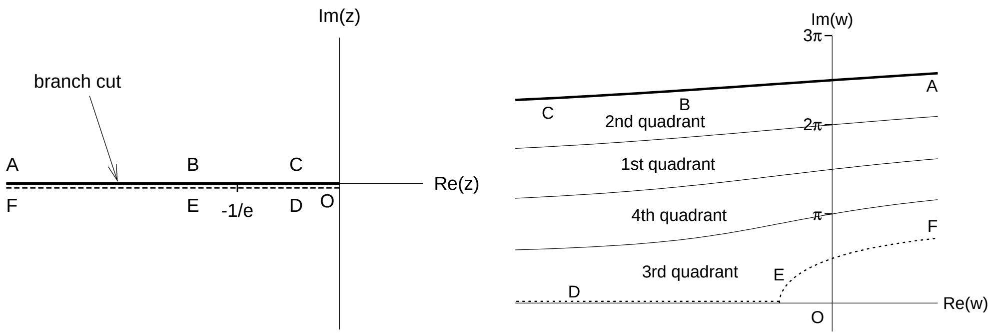

Figure 2.4: Mapping by W_1.

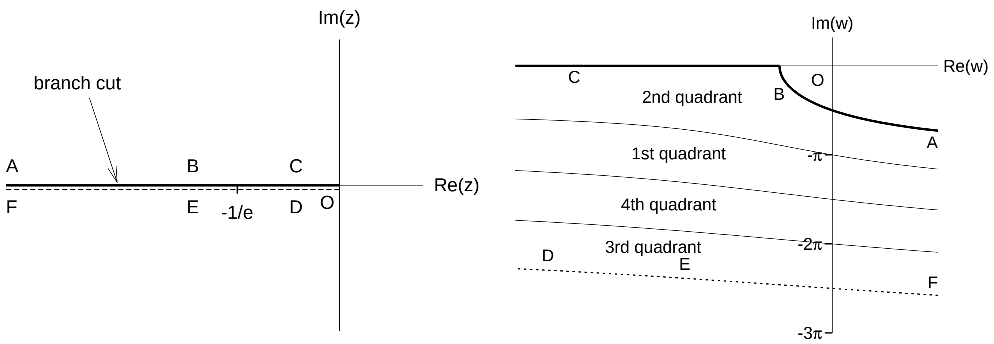

Figure 2.5: Mapping by W_{-1}.

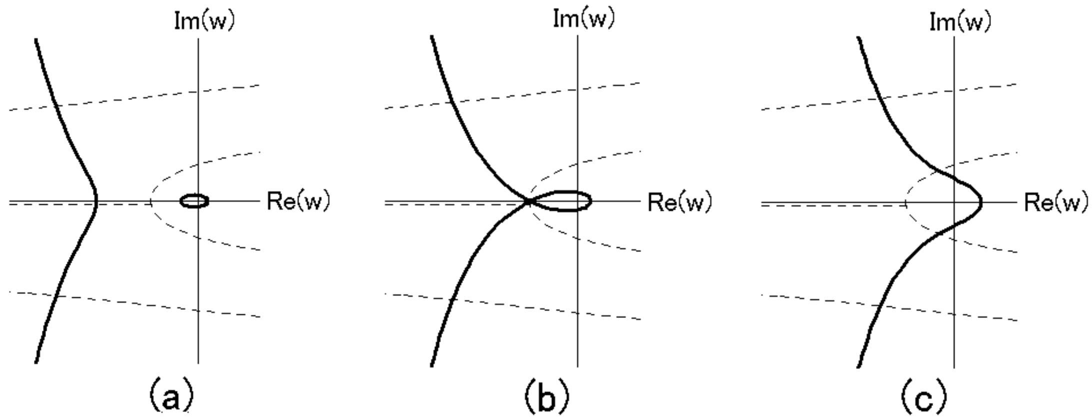

Figure 2.6: $W_k(C_r)$, $k = 0, \pm 1, \pm 2, \cdots$ where $C_r := \{re^{j\theta} \mid \theta \in (-\pi, \pi]$ for (a) $r < -1/e$, (b) $r = -1/e$ and (c) $r > -1/e$.

represents a circle centered the origin with a radius of r for (a) $r < -1/e$, (b) $r = -1/e$ and (c) $r > -1/e$. In the case (a) where C_r has no intersection with B_{C0}, $W_0(C_r)$ is apart from the other branches. Turning to the case (b) where C_r contacts B_{C0}, $W_0(C_r)$ is connected to the other branches. Then they are merged in the case (c) where C_r intersects to B_{C0}. For all the cases, the curve $W_0(C_r)$ is placed in the rightmost side among the all branches; this fact illustrates (2.22). When the intersection of C_r and B_{C0}, say P, which emerges in the case (b) and (c), is mapped, $W_0(P)$ and $W_{-1}(P)$ appear on the upper and lower boundary of W_0 respectively, and thus (2.23) holds.

Lemma 2.3 has been anticipated by a case study in [4], whereas in this thesis, the formal proof for this is given in Appendix A. In the appendix, especially, Lemma A.1 and A.2 are cited in the subsequent discussion, so they should be cast a glance.

2.5.2 Adaptation to Stability Analysis of Linear Time-Delay Systems

In this subsection, underlying theories in the application of the Lambert W function to linear time-delay systems are stated with which the argument of the thesis begins.

First consider a complex-valued linear scalar time-delay system

$$\dot{x}(t) = \alpha x(t) + \beta x(t - \tau), \tag{2.25}$$

where $\alpha, \beta, x(t) \in \mathbf{C}$ and $\tau > 0$ is a time-delay. From Definition 2.2, the system (2.25) is stable if and only if the roots of the characteristic quasi-polynomial

$$p_0(s) := s - \alpha - \beta e^{-\tau s} \tag{2.26}$$

all lie in the complex open left half-plane.

Let us transform the characteristic equation $p_0(s) = 0$ as follows:

$$\begin{aligned} s - \alpha - \beta e^{-\tau s} = 0 \;&\Leftrightarrow\; s - \alpha = \beta e^{-\tau s} \\ &\Leftrightarrow\; (s-\alpha)e^{\tau s} = \beta \\ &\Leftrightarrow\; (s-\alpha)e^{\tau(s-\alpha)} = \beta e^{-\tau\alpha} \\ &\Leftrightarrow\; \tau(s-\alpha)e^{\tau(s-\alpha)} = \tau\beta e^{-\tau\alpha}. \end{aligned} \tag{2.27}$$

Combining (2.27) with (2.17) and (2.18) leads to the following further transformation [4].

$$\begin{aligned} &\tau(s-\alpha) = W(\tau\beta e^{-\tau\alpha}) \\ \Leftrightarrow\;\; & s = \frac{1}{\tau}W(\tau\beta e^{-\tau\alpha}) + \alpha. \end{aligned} \tag{2.28}$$

It should be stressed that (2.28) is an *explicit expression* of the characteristic roots of the system (2.25). The expression (2.28) is favorable for exploring qualitative features of them and the branches of W enable us to identify the positions of all of them in the complex plane at a glance and compute them readily with the help of Mathematica, Maple or Matlab.

The explicit expression (2.28) immediately offers the following stability condition which is fundamental in this thesis.

Lemma 2.4 The linear scalar time-delay system (2.25) is stable if and only if

$$S_W(\alpha,\beta,\tau) := \mathrm{Re}\left[\frac{1}{\tau}W_0(\tau\beta e^{-\tau\alpha}) + \alpha\right] < 0. \tag{2.29}$$

Moreover, the value of $S_W(\alpha,\beta,\tau)$ stands for the stability exponent directly.

Proof The characteristic root $s = W_0(\tau\beta e^{-\tau\alpha})/\tau + \alpha$ is always in the rightmost among all of the characteristic roots $s = W_k(\tau\beta e^{-\tau\alpha})/\tau + \alpha$, $k = 0, \pm 1, \cdots, \pm\infty$ by virtue of Lemma 2.3, so that the lemma follows by Definition 2.2. □

Remark 2.3 In the case (2.23) of Lemma 2.3, as there are two rightmost roots corresponding to W_0 and W_{-1}, either the former or the latter serves as the critical root for stability.

Now return to the linear multivariable time-delay system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{2.30}$$

where $A, B \in \mathbf{C}^{n\times n}$. In order to derive a stability condition for the system (2.30), the following assumption is required.

Assumption 2.1 The characteristic quasi-polynomial of the linear time-delay system (2.30) is equivalent to

$$(s - \alpha_1 - \beta_1 e^{-\tau s}) \cdots (s - \alpha_n - \beta_n e^{-\tau s}), \tag{2.31}$$

where $\alpha_i, \beta_i \in \mathbf{C}$, $i = 1, \cdots, n$.

Remark 2.4 If A and B are *simultaneously triangularizable* for which a necessary and sufficient condition is that the commutator $AB - BA$ is nilpotent [83], then there is a nonsingular matrix that transforms A and B to triangular matrices. Assumption 2.1 is then fulfilled where the diagonal elements of the triangularized matrices are embedded into α_i and β_i, $i = 1, \cdots, n$ in (2.31). In [83], simultaneous triangularization theories are intensively researched. For the multiple time-delay system (2.8), a similar argument can be pursued but a class of the systems that allows the above assumption is terribly limited.

If Assumption 2.1 is made, Lemma 2.4 implies the stability condition for the multivariable system (2.30).

Lemma 2.5 Under Assumption 2.1, the linear time-delay system (2.30) is stable if and only if

$$S_W(\alpha_i, \beta_i, \tau) < 0, \quad i = 1, \cdots, n. \tag{2.32}$$

Moreover, the value of

$$\max_{i=1,\cdots,n} S_W(\alpha_i, \beta_i, \tau) \tag{2.33}$$

stands for the stability exponent.

Furthermore, general robust stability conditions in terms of the Lambert W function can be obtained straightforwardly from Lemmas 2.4 and 2.5. Let Ω^α and Ω^β be compact sets in $\mathbf{C}$ and α, β and τ be uncertain parameters prescribed by

$$\alpha \in \Omega^\alpha, \quad \beta \in \Omega^\beta, \quad \tau \in [\underline{\tau}, \overline{\tau}]. \tag{2.34}$$

Then the following lemma which implies delay-dependently robustness of stability is obvious from Lemma 2.4.

Lemma 2.6 The linear scalar time-delay system (2.25) with the uncertainties prescribed by (2.34) is robustly stable if and only if

$$\max_{\alpha \in \Omega^\alpha, \beta \in \Omega^\beta, \tau \in [\underline{\tau}, \overline{\tau}]} S_W(\alpha, \beta, \tau) < 0 \tag{2.35}$$

Turning to the multivariable system (2.30), under Assumption 2.1, let Ω_i^α and Ω_i^β, $i = 1, \cdots, n$ be compact sets in $\mathbf{C}$ and α_i, β_i, $i = 1, \cdots, n$ and τ be uncertain parameters prescribed by

$$\alpha_i \in \Omega_i^\alpha, \quad \beta_i \in \Omega_i^\beta, \quad i = 1, \cdots, n, \quad \tau \in [\underline{\tau}, \overline{\tau}]. \tag{2.36}$$

The following lemma is again obvious from Lemma 2.5.

Lemma 2.7 Under Assumption 2.1, the linear time-delay system (2.30) with the uncertainties prescribed by (2.36) is robustly stable if and only if

$$\max_{\alpha_i \in \Omega_i^\alpha, \beta_i \in \Omega_i^\beta, \tau \in [\underline{\tau}, \overline{\tau}]} S_W(\alpha_i, \beta_i, \tau) < 0, \quad i = 1, \cdots, n \tag{2.37}$$

Note that Lemmas 2.6 and 2.7 do not reveal where the robust stability is determined in the uncertainties (2.34) or (2.36). If uncertainties are prescribed by suitable forms, the critical part for the robust stability can be specified on extreme points or boundary of the uncertainties. These properties are proven in Chapter 3.

It should be also noticed that Assumption 2.1 is rather restrictive. However, even if Assumption 2.1 cannot be made for given systems, it might be fulfilled again provided that some structural changes of the systems are carried out by feedback controllers. In Chapter 4, it will be shown that *Decoupling* of the systems makes that possible.

The Lambert W function approach is still available for the additional dynamics analysis induced by the model transformations utilized in the Lyapunov approaches. This topic is discussed in Chapter 5.

Chapter 3

Stability Analysis

In this chapter, robust stability of a linear time-delay system

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \tag{3.1}$$

where $A, B \in \mathbf{C}^{n \times n}$, $x(t) \in \mathbf{C}^n$ and $\tau > 0$ is a time-delay, is investigated by the Lambert W function. In practical situations, system modeling might be carried out in the real field rather than complex one. Nevertheless, even for real A and B, if they are triangularized, the resultant triangular matrices possibly have complex elements. Since the Lambert W function puts no border between real-valued and complex-valued systems, if anything, the complex systems might be rather preferable for the robust stability conditions obtained in this chapter. Naturally, the complex-valued systems are more expressive than the real-valued ones. Indeed, behavior of semiconductor lasers in [60] is modeled as complex-valued time-delay systems. Furthermore, if complex-valued neural networks with delay ([104] for delay-free cases) are linearized, then complex-valued linear time-delay systems are required. The more applications of the complex-valued systems are developed, the more powerful the Lambert W function approach will be as expected.

In Section 3.1, *extreme point results* are elucidated with respect to the robust stability of the linear time-delay system (3.1) provided that suitable uncertainties are prescribed. Although the result is confined to the scalar systems, it can be applied to the multivariable ones if the coefficient matrices of (3.1) are simultaneously trianguralizable (see Remark 2.4) or just both of them upper (or lower) triangular matrices.

In Section 3.2, the extreme point results are generalized to *boundary implications* when more flexible uncertainties are assigned. The result is also limited to the scalar systems, simultaneous triangularizability of the system matrices makes it possible to adjust to the multivariable ones again.

In Section 3.3, illustrative examples are presented, giving guidances on the procedures to apply the results in Sections 3.1 and 3.2. Meanwhile, an important technique to measure the time-delay margin guaranteeing the robust stability of the target systems is also shown in this section. Then, this chapter is concluded with some remarks in Section 3.4.

3.1 Extreme Point Results

Consider a scalar linear time-delay system

$$\dot{x}(t) = \alpha x(t) + \beta x(t-\tau), \tag{3.2}$$

where $\alpha = \alpha^R + j\alpha^I$, $\beta = \beta^r e^{j\beta^\theta}$ with $\alpha^R, \alpha^I, \beta^r, \beta^\theta \in \mathbf{R}$, $x(t) \in \mathbf{C}$ and $\tau > 0$ is fixed. Let Ω^α and Ω^β be

$$\begin{aligned} \Omega^\alpha &:= \{\alpha^R + j\alpha^I \mid \alpha^R \in [\underline{\alpha}^R, \overline{\alpha}^R], \alpha^I \in [\underline{\alpha}^I, \overline{\alpha}^I]\}, \\ \Omega^\beta &:= \{\beta^r e^{j\beta^\theta} \mid \beta^r \in [\underline{\beta}^r, \overline{\beta}^r], \beta^\theta \in [\underline{\beta}^\theta, \overline{\beta}^\theta]\}, \end{aligned} \tag{3.3}$$

where $\underline{\alpha}^R \le \overline{\alpha}^R$, $\underline{\alpha}^I \le \overline{\alpha}^I$, $0 \le \underline{\beta}^r \le \overline{\beta}^r$ and $\underline{\beta}^\theta \le \overline{\beta}^\theta$, and suppose that the system (3.2) has uncertainties prescribed by

$$\alpha \in \Omega^\alpha, \quad \beta \in \Omega^\beta. \tag{3.4}$$

The forms of Ω^α and Ω^β are as depicted in Figure 3.1.

In this section, the argument proceeds along a line that finds out $\alpha \in \Omega^\alpha$ and $\beta \in \Omega^\beta$ where $S_W(\alpha, \beta, \tau)$ is maximized for the sake of the connection with Lemma 2.6. In what follows, the variable τ in $S_W(\alpha, \beta, \tau)$ is dropped because τ is a fixed-value and put

$$z_w := \tau\beta e^{-\tau\alpha}. \tag{3.5}$$

Let us reveal monotonicity of $S_W(\alpha, \beta)$ with respect to α^R first.

Lemma 3.1 $S_W(\alpha, \beta)$ is a monotone increasing function of α^R.

Proof For simplicity, $S_W(\alpha, \beta)$ is rewritten as $S_W(\alpha^R)$ because α^I and β can be regarded as constant.

When $\beta = 0$, the lemma is obvious since $S_W(\alpha^R) = \alpha^R$. Assume $\beta \neq 0$. According to the relation between z_w and the branch cut of W_0, separate occasions are dealt with.

(i) The case of $z_w \notin B_{C0} = \{a + j0 \mid -\infty < a \le -1/e\}$.

$W_0(z_w)$ is analytic by Lemma A.1. Differentiating $S_W(\alpha^R)$ with respect to α^R leads to

$$\begin{aligned} \frac{dS_W(\alpha^R)}{d\alpha^R} &= \frac{1}{\tau}\mathrm{Re}\left[\frac{dW_0(z_w)}{dz_w}\frac{dz_w}{d\alpha^R}\right] + \frac{d\alpha^R}{d\alpha^R} \\ &= \mathrm{Re}\left[-\frac{W_0(z_w)}{1 + W_0(z_w)}\right] + 1, \end{aligned} \tag{3.6}$$

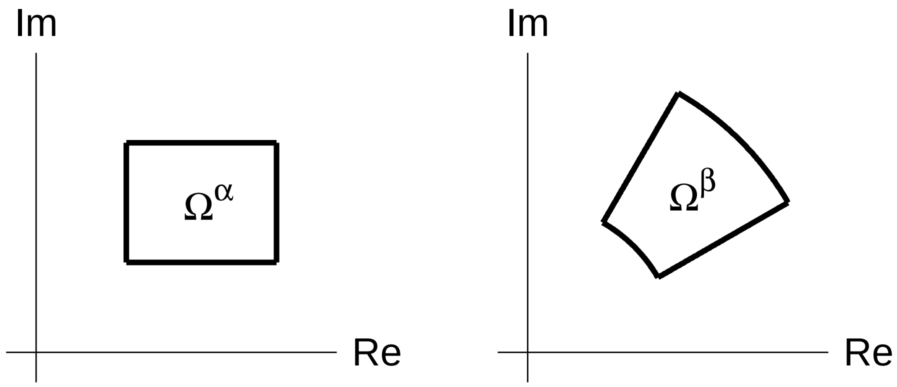

Figure 3.1: Forms of Ω^α and Ω^β in (3.4).

which is written as

$$-\frac{\xi_0(1+\xi_0)+\eta_0^2}{(1+\xi_0)^2+\eta_0^2}+1 \tag{3.7}$$

by setting $W_0(z_w) = \xi_0 + j\eta_0$. The fact that $\xi_0 > -1$ (since $z_w \neq -1/e$) implies (3.7)>0, i.e. $dS_W(\alpha^R)/d\alpha^R > 0$. This proves that $S_W(\alpha^R)$ is a monotone increasing function of α^R for $z_w \notin B_{C0}$.

(ii) The case of $z_w \in \tilde{B}_{C0} := \{a + j0 \mid -\infty < a < -1/e\}$.

Defining a function

$$W_{01}(z) := \begin{cases} W_0(z), & \mathrm{Im}(z) \geq 0 \\ W_1(z), & \mathrm{Im}(z) < 0 \end{cases}, \tag{3.8}$$

(3.8) is analytic in $\tilde{B}_{C0}$. Letting $\tilde{S}_W(\alpha^R) := \mathrm{Re}\left[W_{01}(z_w)\right]/\tau + \alpha^R$,

$$\frac{d\tilde{S}_W(\alpha^R)}{d\alpha^R} = \mathrm{Re}\left[-\frac{W_{01}(z_w)}{1+W_{01}(z_w)}\right]+1 \tag{3.9}$$

holds and $d\tilde{S}_W(\alpha^R)/d\alpha^R > 0$ is fulfilled similarly to the case (i). This shows that $\tilde{S}_W(\alpha^R)$ is again monotone increasing with respect to α^R, so that $S_W(\alpha^R)$ is too since $\tilde{S}_W(\alpha^R) = S_W(\alpha^R)$ for $z_w \in \tilde{B}_{C0}$.

(iii) The case of $z_w = -1/e$.

Finally, consider the behavior of $S_W(\alpha^R)$ at α^R such that $z_w = \tau\beta e^{-\tau(\alpha^R + j\alpha^I)} = -1/e$. Noting that z_w varies radially along the lines centered at the origin according to α^R, it is true that $z_w = -1/e$ only if the argument of z_w remains π for arbitrary α^R. This allows to fix one's eyes to the negative real axis without loss of generality. The cases

(i) and (ii) suggest that $S_W(\alpha^R)$ is a monotone increasing function with respect to α^R in the neighborhood of $z = -1/e$. Futhermore, $S_W(\alpha^R)$ is continuous at α^R satisfying $z_w = -1/e$ and hence $S_W(\alpha^R)$ is non-decreasing as well at this point. □

Here switch to consider β^r. As preliminaries, let us assign variables to the domain and the range of the Lambert W function (2.18) as

$$z = a + jb, \quad w = \xi + j\eta. \tag{3.10}$$

Substituting (3.10) into (2.17) gives the equations

$$a = e^{\xi}(\xi \cos\eta - \eta \sin\eta), \tag{3.11}$$
$$b = e^{\xi}(\eta \cos\eta + \xi \sin\eta). \tag{3.12}$$

To identify the maximal points of $S_W(\alpha, \beta)$ with respect to β^r, consider the image of a line segment

$$S_g := \{p(c + jd) \mid p \in [\underline{p}, \overline{p}]\}, \tag{3.13}$$

by W_0, where $c, d \in \mathbf{R}$ and $\underline{p} < \overline{p}$. Then

$$cp = e^{\xi}(\xi \cos\eta - \eta \sin\eta), \tag{3.14}$$
$$dp = e^{\xi}(\eta \cos\eta + \xi \sin\eta) \tag{3.15}$$

follow from (3.11) and (3.12). The aim in the subsequent paragraphs is to prove that the maximum of $\mathrm{Re}[\,W_0(S_g)\,]$ is taken at either of the extreme points of the segment (3.13).

Suppose that $c \neq 0$ and $d \neq 0$. Eliminating p from (3.14) and (3.15) generates

$$\xi = \frac{d \tan\eta + c}{d - c \tan\eta}\eta. \tag{3.16}$$

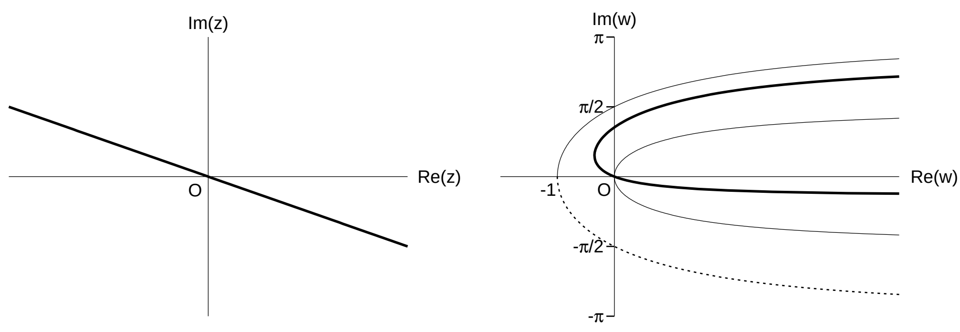

Figure 3.2: S_g in the z-plane and $W_0(S_g)$ in the w-plane.

For (3.16), the principal branch W_0 corresponds to

$$\xi = \frac{d\tan\eta + c}{d - c\tan\eta}\eta, \quad \underline{\eta} < \eta < \overline{\eta}, \tag{3.17}$$

where $\underline{\eta} = \tan^{-1}(d/c) - \pi$ and $\overline{\eta} = \tan^{-1}(d/c)$ under the assumption $\tan^{-1}(\cdot) \in (0, \pi)$. Figure 3.2 shows the correspondence of the line (3.13) with the image mapped by W_0.

In this paragraph, it is proven that (3.17) is leftward convex. Differentiating ξ with respect to η gives

$$\frac{d\xi}{d\eta} = \frac{(c^2+d^2)(1+\tan^2\eta)\eta}{(d-c\tan\eta)^2} + \frac{d\tan\eta + c}{d - c\tan\eta}, \tag{3.18}$$

and differentiating $d\xi/d\eta$ further gives

$$\frac{d^2\xi}{d\eta^2} = \frac{2(c^2+d^2)(1+\tan^2\eta)}{(d-c\tan\eta)^2}(\xi+1). \tag{3.19}$$

It is realized from (3.18) and (3.19) that $d\xi/d\eta < 0$ and $d\xi/d\eta > 0$ for $\eta = \tan^{-1}(d/c+\epsilon) - \pi$ and $\eta = \tan^{-1}(d/c - \epsilon)$ with sufficiently small $\epsilon > 0$ respectively and $d^2\xi/d\eta^2 > 0$ since $\xi > -1$, i.e. $d\xi/d\eta$ is monotone increasing for $\eta \in (\underline{\eta}, \overline{\eta})$. According to the Intermediate Value Theorem and the monotonicity of $d\xi/d\eta$, there exists a $\eta \in (\underline{\eta}, \overline{\eta})$ such that $d\xi/d\eta = 0$. These facts indicate the graph of (3.17) is leftward convex.

Put points $\underline{z} = \underline{p}(c + jd)$ and $\overline{z} = \overline{p}(c + jd)$ where on the segment (3.13), and suppose that contrary to the claim, the maximum of $\mathrm{Re}[\,W_0(S_g)\,]$ is taken at a middle point $z_0 = p_0(c+jd)$, $p_0 \in (\underline{p}, \overline{p})$ between $\underline{z}$ and $\overline{z}$. Then, $W_0(\underline{z})$ and $W_0(\overline{z})$ lie on more left than $W_0(z_0)$ in the w-plane. However, since $W_0(S_g)$ is a continuous curve and bijective (Remark 2.2) and the leftward-convexity implies the right side of the graph is open, the curve $W_0(S_g)$ must be overlapped in a certain interval. This is contradictory to the fact that the mapping is bijective. Namely, the maximum of $\mathrm{Re}[\,W_0(z)\,]$ must be taken at either $\underline{z}$ or $\overline{z}$ (see Figure 3.3).

In the case of $c = 0$ and $d \neq 0$ which are the extreme cases of the previous one, $W_0(S_g)$ again forms a leftward-convex continuous curve and W_0 is proven to be bijective. Therefore, the similar argument can be repeated, leading to the same conclusion (see Figure 3.4).

The case of $c \neq 0$ and $d = 0$ has one singularity at the point $W_0(-1/e) = -1$. Nevertheless, the similar argument can be maintained as conjectured by the shape of $W_0(S_g)$ and the fact that $W_0(S_g)$ is a continuous curve and bijective (see Figure 3.5).

The above discussion yields the following lemma, which elucidates the maximal points of $S_W(\alpha, \beta)$ with respect to β^r.

Lemma 3.2 Let α and β^θ be constant. Then,

$$\max_{\beta^r \in [\underline{\beta}^r, \overline{\beta}^r]} S_W(\alpha, \beta^r e^{j\beta^\theta}) = \max\left\{S_W(\alpha, \underline{\beta}^r e^{j\beta^\theta}),\ S_W(\alpha, \overline{\beta}^r e^{j\beta^\theta})\right\}. \tag{3.20}$$

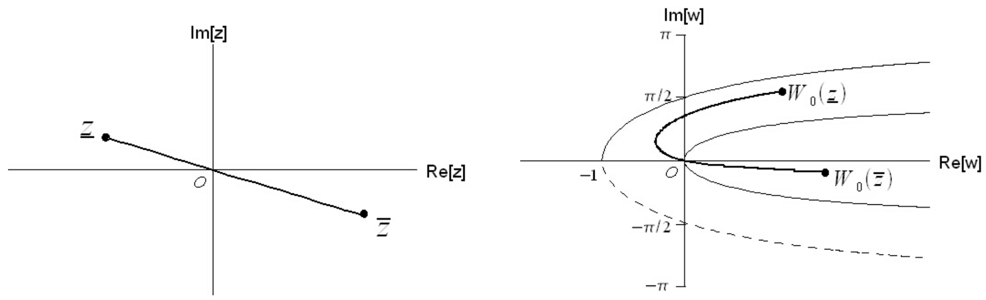

Figure 3.3: Mapping of S_g by W_0 in the case of $c \neq 0$ and $d \neq 0$.

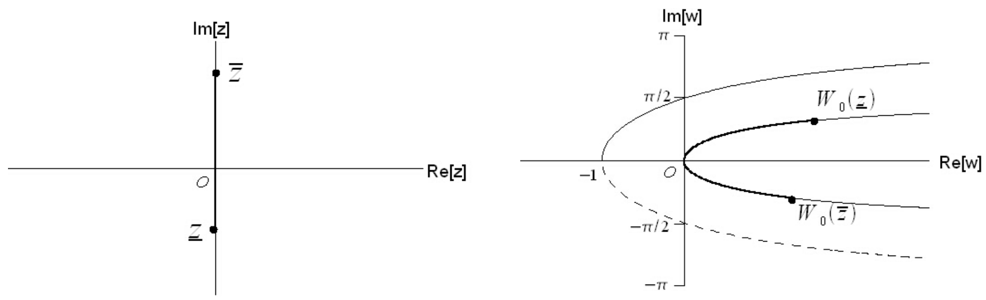

Figure 3.4: Mapping of S_g by W_0 in the case of $c = 0$ and $d \neq 0$.

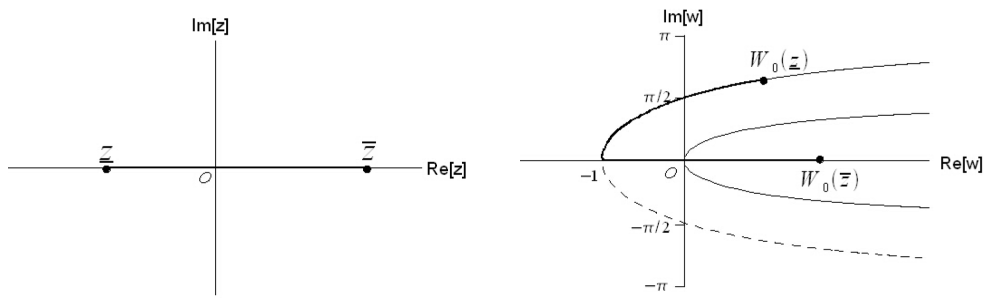

Figure 3.5: Mapping of S_g by W_0 in the case of $c \neq 0$ and $d = 0$.

Proof As β^r varies in $[\underline{\beta}^r, \overline{\beta}^r]$, z_w moves along a segment similar to (3.13) and the extreme points of its locus correspond to $\underline{\beta}^r$ and $\overline{\beta}^r$. The lemma follows from the above discussion. $\square$

Finally, it remains to consider α^I and β^θ. Both of them are embedded into the argument of z_w. Therefore they can be treated like as one variable. The following lemma clarifies the remaining maximal points of $S_W(\alpha, \beta)$ with respect to α^I and β^θ.

Lemma 3.3 Let α^R and β^r be constant and define

$$C^{\alpha\beta} := \left\{ e^{j(\beta^\theta - \tau\alpha^I)} \;\middle|\; \alpha^I \in [\underline{\alpha}^I, \overline{\alpha}^I], \beta^\theta \in [\underline{\beta}^\theta, \overline{\beta}^\theta] \right\}. \tag{3.21}$$

(I) If $C^{\alpha\beta}$ crosses the positive real axis,

$$\max_{\alpha^I \in [\underline{\alpha}^I, \overline{\alpha}^I], \beta^\theta \in [\underline{\beta}^\theta, \overline{\beta}^\theta]} S_W(\alpha, \beta) = S_W(\alpha^R, \beta^r). \tag{3.22}$$

(II) If $C^{\alpha\beta}$ does not cross the positive real axis,

$$\max_{\alpha^I \in [\underline{\alpha}^I, \overline{\alpha}^I], \beta^\theta \in [\underline{\beta}^\theta, \overline{\beta}^\theta]} S_W(\alpha, \beta) = \max \left\{ S_W(\alpha^R + j\underline{\alpha}^I, \beta^r e^{j\overline{\beta}^\theta}), S_W(\alpha^R + j\overline{\alpha}^I, \beta^r e^{j\underline{\beta}^\theta}) \right\}. \tag{3.23}$$

Proof When $\beta^r = 0$, the lemma is obvious since $S_W(\alpha, \beta) = \alpha^R$. Let $\beta^r > 0$. Define

$$C_{z_w}^{\alpha\beta} := \left\{ z_w \;\middle|\; \alpha^I \in [\underline{\alpha}^I, \overline{\alpha}^I], \beta^\theta \in [\underline{\beta}^\theta, \overline{\beta}^\theta] \right\}. \tag{3.24}$$

Note that $C_{z_w}^{\alpha\beta}$ is either an arc or a circle and $C^{\alpha\beta}$ is the argument of $C_{z_w}^{\alpha\beta}$.

(I) The case where $C^{\alpha\beta}$ crosses the positive real axis.

Lemma A.2 claims that the maximum of $\mathrm{Re}[\, W_0(z_w)\,]$ on $C_{z_w}^{\alpha\beta}$ is taken at the intersection of $C_{z_w}^{\alpha\beta}$ with the positive real axis. Noting that this intersection occurs at $z_w = \tau\beta^r e^{-\tau\alpha^R}$, the lemma holds.

(II) The case where $C^{\alpha\beta}$ does not cross the positive real axis.

Since $C_{z_w}^{\alpha\beta}$ does not represent a full circle, i.e. $C_{z_w}^{\alpha\beta}$ is an arc, it has two candidate extreme points corresponding to $\{\underline{\alpha}^I, \overline{\beta}^\theta\}$ and $\{\overline{\alpha}^I, \underline{\beta}^\theta\}$. Lemma A.2 says that the maximum of $\mathrm{Re}(W_0(z_w))$ on $C_{z_w}^{\alpha\beta}$ is taken at either of those extreme points. $\square$

Remark 3.1 Figure 3.6 shows the positions of the crucial points in $C^{\alpha\beta}$ of Lemma 3.3 for the robust stability with respect to α^I and β^θ.

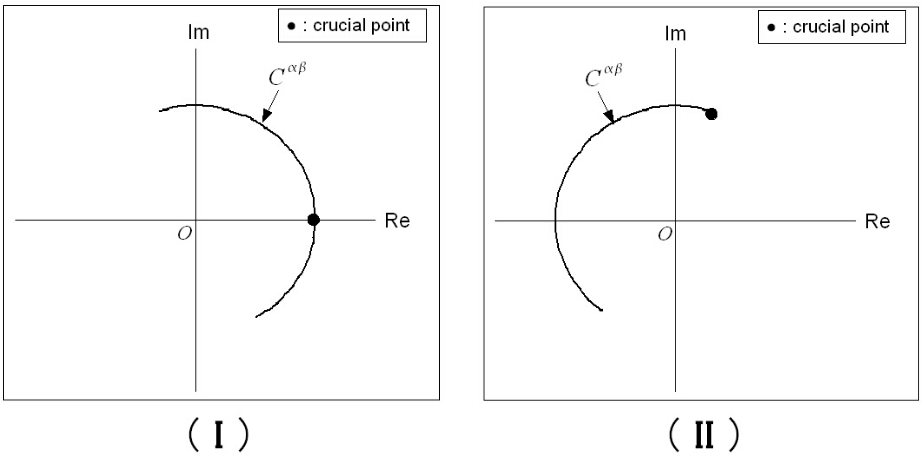

Figure 3.6: The crucial points for the robust stability in $C^{\alpha\beta}$: the cases (I) and (II) of Lemma 3.3.

Combining Lemmas 3.1, 3.2 and 3.3 yields the following *extreme point result* on the robust stability of the scalar system (3.2) with the uncertainties (3.4).

Theorem 3.4 The scalar linear time-delay system (3.2) with the uncertainties prescribed by (3.4) is robustly stable for the fixed time-delay $\tau > 0$ if and only if

(I) for $C^{\alpha\beta}$ crossing the positive real axis,

$$\max\left\{S_W(\overline{\alpha}^R, \underline{\beta}^r),\ S_W(\overline{\alpha}^R, \overline{\beta}^r)\right\} < 0. \tag{3.25}$$

(II) for $C^{\alpha\beta}$ not crossing the positive real axis,

$$\begin{aligned}\max\Big\{&S_W(\overline{\alpha}^R + j\underline{\alpha}^I, \underline{\beta}^r e^{j\overline{\beta}^\theta}), S_W(\overline{\alpha}^R + j\underline{\alpha}^I, \overline{\beta}^r e^{j\overline{\beta}^\theta}),\\ &S_W(\overline{\alpha}^R + j\overline{\alpha}^I, \underline{\beta}^r e^{j\underline{\beta}^\theta}), S_W(\overline{\alpha}^R + j\overline{\alpha}^I, \overline{\beta}^r e^{j\underline{\beta}^\theta})\Big\} < 0. \end{aligned}\tag{3.26}$$

Proof Lemmas 3.1, 3.2 and 3.3 tell where the maximal points of $S_W(\alpha, \beta)$ with respect to α^R, β^r and α^I-β^θ are taken in the uncertainties (3.4) respectively. In this way, the theorem associated with Lemma 2.6 is true. □

Figures 3.7 and 3.8 show the critical points for the robust stability of the system (3.2) with (3.4) at which Theorem 3.4 is satisfied. Accordingly, extreme point results hold true concerning the robust stability of (3.2) if the regions for uncertainties in the coefficients are

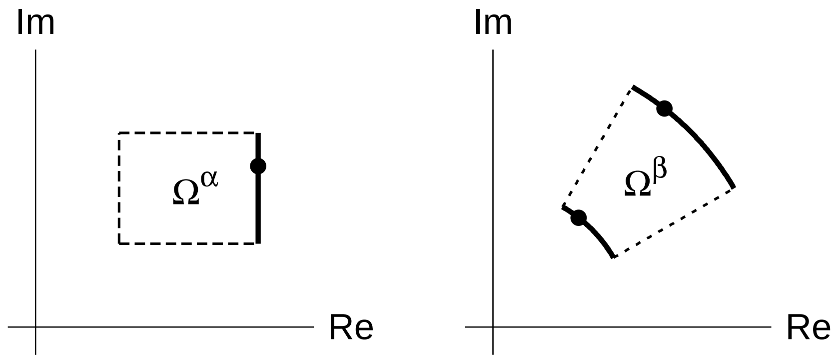

Figure 3.7: Points determining robust stability in the case (I) of Theorem 3.4: they lie on the solid line and their locations depend on the delay τ.

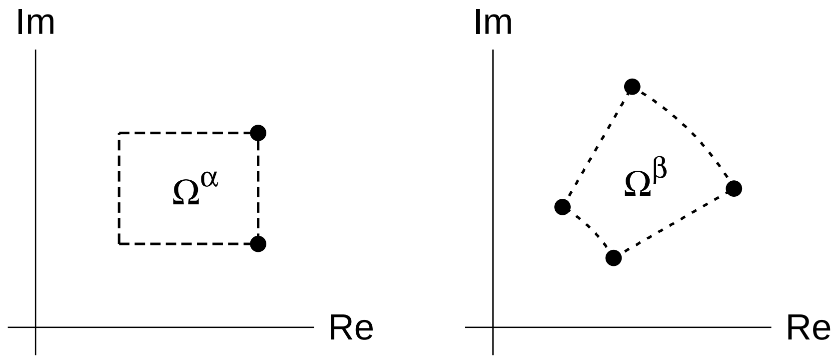

Figure 3.8: Points determining robust stability in the case (II) of Theorem 3.4.

suitably prescribed: for the non-delay term an axis-parallel box and for the delay term a sector form.

Now change the subject to the multivariable system (3.1). Put Assumption 2.1 on the system (3.1) and let α_i and β_i, $i = 1, \cdots, n$ in (2.31) be uncertain parameters prescribed by

$$\alpha_i \in \Omega_i^\alpha, \quad \beta_i \in \Omega_i^\beta, \quad i = 1, \cdots, n, \tag{3.27}$$

where

$$\begin{aligned} \Omega_i^\alpha &:= \{\alpha_i^R + j\alpha_i^I \mid \alpha_i^R \in [\underline{\alpha}_i^R, \overline{\alpha}_i^R], \alpha_i^I \in [\underline{\alpha}_i^I, \overline{\alpha}_i^I]\}, \quad i = 1, \cdots, n \\ \Omega_i^\beta &:= \{\beta_i^r e^{j\beta_i^\theta} \mid \beta_i^r \in [\underline{\beta}_i^r, \overline{\beta}_i^r], \beta_i^\theta \in [\underline{\beta}_i^\theta, \overline{\beta}_i^\theta]\}, \quad i = 1, \cdots, n \end{aligned} \tag{3.28}$$

where $\underline{\alpha}_i^R \le \overline{\alpha}_i^R$, $\underline{\alpha}_i^I \le \overline{\alpha}_i^I$, $0 \le \underline{\beta}_i^r \le \overline{\beta}_i^r$ and $\underline{\beta}_i^\theta \le \overline{\beta}_i^\theta$, $i = 1, \cdots, n$.

Theorem 3.4 provides the following robust stability condition for the system (3.1) with the uncertainties (3.27).

Corollary 3.5 Under Assumption 2.1, the linear time-delay system (3.1) whose characteristic quasi-polynomial is (2.31) with the uncertainties prescribed by (3.27) is robustly stable for the fixed time-delay $\tau > 0$ if and only if given

$$C_i^{\alpha\beta} := \left\{ e^{j(\beta_i^\theta - \tau\alpha_i^I)} \;\middle|\; \alpha_i^I \in [\underline{\alpha}_i^I, \overline{\alpha}_i^I], \beta_i^\theta \in [\underline{\beta}_i^\theta, \overline{\beta}_i^\theta] \right\}, \quad i = 1, \cdots, n, \tag{3.29}$$

(I) for $C_i^{\alpha\beta}$, $i = 1, \cdots, n$ crossing the positive real axis,

$$\max\left\{ S_W(\overline{\alpha}_i^R, \underline{\beta}_i^r),\ S_W(\overline{\alpha}_i^R, \overline{\beta}_i^r) \right\} < 0, \quad i = 1, \cdots, n \tag{3.30}$$

are satisfied respectively.

(II) for $C_i^{\alpha\beta}$, $i = 1, \cdots, n$ not crossing the positive real axis,

$$\begin{aligned} \max\Big\{ & S_W(\overline{\alpha}_i^R + j\underline{\alpha}_i^I, \underline{\beta}_i^r e^{j\overline{\beta}_i^\theta}), S_W(\overline{\alpha}_i^R + j\underline{\alpha}_i^I, \overline{\beta}_i^r e^{j\overline{\beta}_i^\theta}), \\ & S_W(\overline{\alpha}_i^R + j\overline{\alpha}_i^I, \underline{\beta}_i^r e^{j\underline{\beta}_i^\theta}), S_W(\overline{\alpha}_i^R + j\overline{\alpha}_i^I, \overline{\beta}_i^r e^{j\underline{\beta}_i^\theta}) \Big\} < 0, \\ & \qquad\qquad i = 1, \cdots, n \end{aligned} \tag{3.31}$$

are satisfied respectively.

Proof It is obvious from Theorem 3.4 and Lemma 2.7. □

3.2 Boundary Implications

Let us reconsider scalar linear time-delay system (3.2), where, however the uncertainties are modified into

$$\alpha \in \hat{\Omega}^\alpha, \quad \beta \in \hat{\Omega}^\beta, \tag{3.32}$$

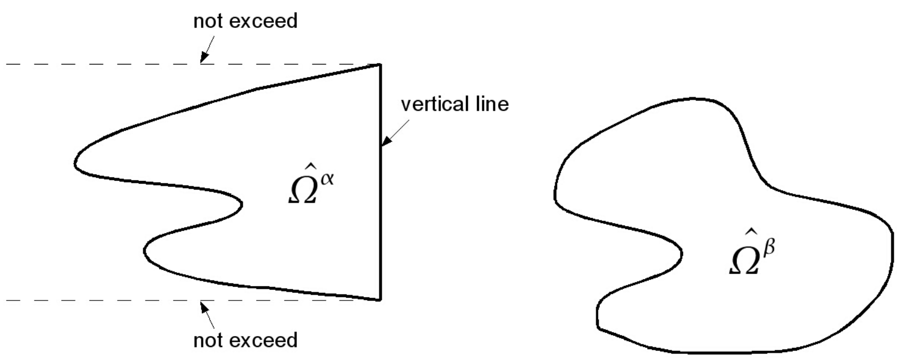

Figure 3.9: Forms of $\hat{\Omega}^\alpha$ and $\hat{\Omega}^\beta$ in (3.32).

where $\hat{\Omega}^{\alpha}$ is a compact set whose real-valued maximums are identical for all imaginary parts and $\hat{\Omega}^{\beta}$ is an arbitrary compact set as depicted in Figure 3.9. Clearly, $\hat{\Omega}^{\alpha}$ and $\hat{\Omega}^{\beta}$ are more general than Ω^{α} and Ω^{β} in (3.3). The goal of this subsection is to prove *boundary implications* of stability, the property that such a stability property as robust stability is determined on the boundaries of the uncertainties, of the system (3.2) with the uncertainties (3.32). It is notable that the results of this section is generalizations of the previous extreme point results.

The argument is begun with an elemental result of complex functions which is an analogy to the Maximum Principle [3].

Lemma 3.6 Let Ω be a nonempty connected open set in the complex plane. Suppose $f(z)$ is an analytic function in Ω and $\mathrm{Re}[\,f(z)\,]$ is continuous in the closure $\overline{\Omega}$. Then the maximum of $\mathrm{Re}[\,f(z)\,]$ on $\overline{\Omega}$ is taken on the boundary $\partial\Omega$.

Proof The proof proceeds in the same way as that of the Maximum Principle [3] regarding to the modulus of $f(z)$. $\square$

The next lemma is needed to prove that Lemma 3.6 also holds true for the Lambert W function.

Lemma 3.7 $\mathrm{Re}[\,W_0(z)\,]$ is monotone decreasing in B_{C0}. $\mathrm{Re}[\,W_k(z)\,]$, $k = \pm 1, \cdots, \pm\infty$ are monotone decreasing in $\tilde{B}_C := B_C \setminus \{0\}$.

Proof Fisrt note in the definition of $\tilde{B}_C$ that the zero is excluded from B_C. The reason is that $\mathrm{Re}[\,W_k(z)\,]$, $k = \pm 1, \cdots, \pm\infty$ diverge to $-\infty$ as $z \to 0$.

The images of the branch cuts, $W_0(B_{C0})$ and $W_k(\tilde{B}_C)$, $k = \pm 1, \cdots, \pm\infty$ can be represented in terms of w-variables ξ and η in (3.10) as follows:

$$W_0(B_{C0}) = \left\{ \frac{-\eta}{\tan\eta} + j\eta \;\middle|\; \eta \in [0, \pi) \right\}, \tag{3.33}$$

$$W_k(\tilde{B}_C) = \left\{ \frac{-\eta}{\tan\eta} + j\eta \;\middle|\; \eta \in (2k\pi, (2k+1)\pi) \right\}, \quad k = 1, \cdots, \infty, \tag{3.34}$$

$$W_{-1}(\tilde{B}_C) = \left\{ \frac{-\eta}{\tan\eta} + j\eta \;\middle|\; \eta \in (-\pi, 0] \right\} \bigcup \{\xi + j0 \mid \xi \in (-\infty, -1)\}, \tag{3.35}$$

$$W_k(\tilde{B}_C) = \left\{ \frac{-\eta}{\tan\eta} + j\eta \;\middle|\; \eta \in ((2k+1)\pi, (2k+2)\pi) \right\}, \quad k = -2, \cdots, -\infty. \tag{3.36}$$

Note that $\{-\eta/\tan\eta + j\eta \mid \eta \in (-\pi, 0]\}$ and $\{\xi + j0 \mid \xi \in (-\infty, -1)\}$ in (3.35) correspond to $W_{-1}(B_{C0})$ and $W_{-1}(\tilde{B}_{C1})$ where $\tilde{B}_{C1} := \tilde{B}_C \setminus B_{C0}$ respectively.

Substituting $\xi = -\eta/\tan\eta$ into (3.11) yields

$$\begin{aligned} a(\eta) &= e^{\frac{-\eta}{\tan\eta}}\left(-\frac{\eta}{\tan\eta}\cos\eta - \eta\sin\eta\right) \\ &= e^{\frac{-\eta}{\tan\eta}}\left(-\frac{\eta\cos^2\eta + \eta\sin^2\eta}{\sin\eta}\right) \\ &= -e^{\frac{-\eta}{\tan\eta}}\frac{\eta}{\sin\eta}. \end{aligned} \tag{3.37}$$

Differentiating (3.37) with respect to η further provides

$$\begin{aligned} \frac{da}{d\eta} &= -e^{\frac{-\eta}{\tan\eta}}\frac{-\tan\eta + \frac{\eta}{\cos^2\eta}}{\tan^2\eta}\cdot\frac{\eta}{\sin\eta} - e^{\frac{-\eta}{\tan\eta}}\frac{\sin\eta - \eta\cos\eta}{\sin^2\eta} \\ &= -e^{\frac{-\eta}{\tan\eta}}\left(-\frac{\eta\cos\eta}{\sin^2\eta} + \frac{\eta^2}{\sin^3\eta} + \frac{1}{\sin\eta} - \frac{\eta\cos\eta}{\sin^2\eta}\right) \\ &= -\frac{e^{\frac{-\eta}{\tan\eta}}}{\sin^3\eta}(-2\eta\cos\eta\sin\eta + \eta^2 + \sin^2\eta) \\ &= -\frac{e^{\frac{-\eta}{\tan\eta}}}{\sin^3\eta}(-\eta\sin 2\eta + \eta^2 + \sin^2\eta). \end{aligned} \tag{3.38}$$

Defining

$$\gamma(\eta) = -\eta\sin 2\eta + \eta^2 + \sin^2\eta \tag{3.39}$$

and differentiating $\gamma(\eta)$ with respect to η gives

$$\begin{aligned} \frac{d\gamma}{d\eta} &= -\sin 2\eta - 2\eta\cos 2\eta + 2\eta + \sin 2\eta \\ &= 2\eta(1-\cos 2\eta). \end{aligned} \tag{3.40}$$

For $\eta \in (2k\pi, (2k+1)\pi)$, $k = 0, 1, \cdots, \infty$, $\gamma(\eta) > 0$ is true because $d\gamma/d\eta > 0$ and $\gamma(0) = 0$ and hence we have $da/d\eta < 0$. Since $a(\eta)$ is continuous at $\eta = 0$ from (3.37), it follows that $a(\eta)$ is a monotone decreasing fucntion of $\eta \in [0, \pi)$ and $\eta \in (2k\pi, (2k+1)\pi)$, $k = 1, \cdots, \infty$. This observation proves that $\eta(a)$, which is the inverse function of $a(\eta)$, is monotone decreasing in $a \in B_{C0}$ when B_{C0} is mapped by W_0 and $a \in \tilde{B}_C$ when $\tilde{B}_C$ is mapped by W_k, $k = 1, \cdots, \infty$.

For $\eta \in ((2k+1)\pi, (2k+2)\pi)$, $k = -1, \cdots, -\infty$, $\gamma(\eta) > 0$ is verified from the fact that $d\gamma/d\eta < 0$ and $\gamma(0) = 0$, so $da/d\eta > 0$. Similarly we can show that $a(\eta)$ is monotone increasing in $\eta \in (-\pi, 0]$ and $\eta \in ((2k+1)\pi, (2k+2)\pi)$, $k = -2, \cdots, -\infty$. This time, it is justifies that $\eta(a)$ is monotone increasing in $a \in B_{C0}$ when B_{C0} is mapped by W_{-1} and $a \in \tilde{B}_C$ when $\tilde{B}_C$ is mapped by W_k, $k = -2, \cdots, -\infty$.

Furthermore, $\xi = -\eta/\tan\eta$ is monotone increasing in each interval of $\eta \in (2k\pi, (2k+1)\pi)$, $k = 0, 1, \cdots, \infty$ and monotone decreasing in $\eta \in ((2k+1)\pi, (2k+2)\pi)$, $k =$

$-1, \cdots, -\infty$. In this way, it turns out that $\xi(a)$ is a monotone decreasing function of $a \in B_{C0}$ in the case where B_{C0} is mapped by W_0 and W_{-1}, i.e. both of $\mathrm{Re}[\,W_0(z)\,]$ and $\mathrm{Re}[\,W_{-1}(z)\,]$ are in $z \in B_{C0}$, and $a \in \tilde{B}_C$ in the other case where $\tilde{B}_C$ is mapped W_k, $k = 1, \pm 2, \cdots, \pm\infty$, i.e. $\mathrm{Re}[\,W_0(z)\,]$, $k = 1, \pm 2, \cdots, \pm\infty$ are in $z \in \tilde{B}_C$.

In the rest of the proof, we deal with the remaining case where $\tilde{B}_{C1}$ is mapped by W_{-1} in which $W_{-1}(\tilde{B}_{C1}) = \{\xi + j0 \mid \xi \in (-\infty, -1)\}$. Substituting $\eta = 0$ into (3.11), we have

$$a(\xi) = \xi e^{\xi}. \tag{3.41}$$

Differentiating (3.41) with respect to ξ gives

$$\frac{da}{d\xi} = e^{\xi}(1 + \xi). \tag{3.42}$$

It is easy to see that $da/d\xi < 0$ for $\xi \in (-\infty, -1)$. Therefore, $a(\xi)$ in (3.41) is monotone decreasing in $\xi \in (-\infty, -1)$, that is, the inverse function $\xi(a)$ is monotone decreasing in $a \in \tilde{B}_{C1}$, in other words $\mathrm{Re}[\,W_{-1}(z)\,]$ is in $z \in \tilde{B}_{C1}$. The proof is now completed. $\Box$

Now the Maximum Principle for $\mathrm{Re}[W_k(z)]$, $k = 0, \pm 1, \cdots, \pm\infty$ can be elucidated by Lemmas A.1, 3.6 and 3.7.

Lemma 3.8 Let Ω be a region as in Lemma 3.6. The maximum of $\mathrm{Re}[\,W_k(z)\,]$, $k = 0, \pm 1, \cdots, \pm\infty$ on $\overline{\Omega}$ is taken on $\partial\Omega$.

Proof The proof can be carried out in the same way for each of $\mathrm{Re}[\,W_k(z)\,]$, $k = 0, \pm 1, \cdots, \pm\infty$. Thus only the W_0 case is proven. Discussions for the other branches can be done by replacing W_0 and B_{C0} with W_k's and B_C respectively.

(i) The case where Ω does not include any part of B_{C0}.

Lemma A.1 proves that $W_0(z)$ is analytic in Ω. Furthermore, $\mathrm{Re}[\,W_0(z)\,]$ is continuous in $\overline{\Omega}$ (see Remark 2.2). Then, Lemma 3.6 readily gives the result.

(ii) The case where Ω includes some part of B_{C0}.

In this case, $W_0(z)$ is not analytic in Ω because $W_0(z)$ is not differentiable in terms of complex functions at $z \in B_{C0}$. However, Ω can be separated by B_{C0} into several sub-regions where $W_0(z)$ can be analytic. Let these sub-regions be $\Omega_1, \cdots, \Omega_m$, $m \geq 1$ (for some typical situations, see Figure 3.10). Note that the boundary of each $\Omega_1, \cdots, \Omega_m$ corresponds to $\partial\Omega$ and B_{C0}. As in the case (i), $W_0(z)$ is analytic in each $\Omega_1, \cdots, \Omega_m$ and $\mathrm{Re}[\,W_0(z)\,]$ is continuous in each $\overline{\Omega}_1, \cdots, \overline{\Omega}_m$. Therefore, Lemma 3.6 implies that the maximum of $\mathrm{Re}[\,W_0(z)\,]$ on each $\overline{\Omega}_1, \cdots, \overline{\Omega}_m$ is taken on $\partial\Omega_1, \cdots, \partial\Omega_m$ respectively, that is, the maximum of $\mathrm{Re}[\,W_0(z)\,]$ on $\overline{\Omega}$ is taken on either $\partial\Omega$ or B_{C0}.

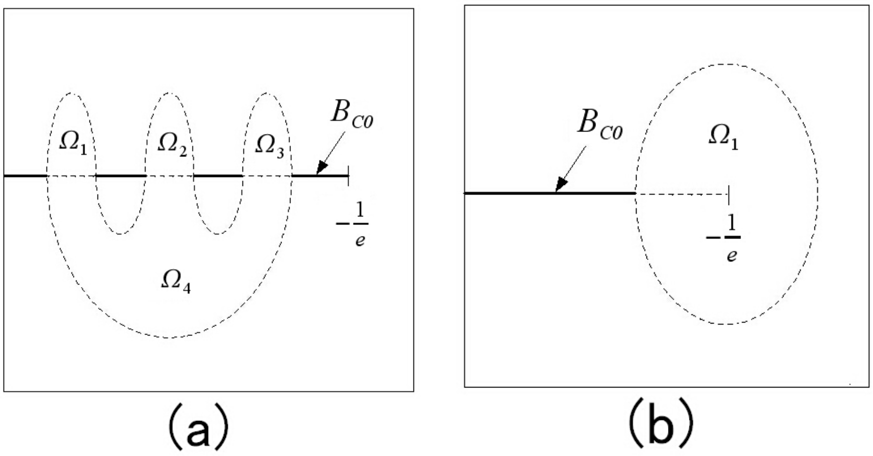

Figure 3.10: Typical decomposition of Ω: (a) $m = 4$. (b) $m = 1$.

Now because of Lemma 3.7, it turns out that the maximum of $\mathrm{Re}[\,W_0(z)\,]$ on B_{C0} is taken at the leftmost point. Moreover, B_{C0} intersects with $\partial\Omega$ at the leftmost point since Ω is bounded and B_{C0} infinitely extends leftward in the complex plane. Hence, we can conclude that the maximum of $\mathrm{Re}[\,W_0(z)\,]$ on $\overline{\Omega}$ is taken on $\partial\Omega$. $\square$

From Lemmas 2.6, 3.1 and 3.8, a boundary implication of stability of the system (3.2) with the uncertainties (3.32) holds.

Theorem 3.9 Define the right edge of $\hat{\Omega}^\alpha$ as

$$\partial^r\hat{\Omega}^\alpha := \{\overline{\alpha}^R + j\alpha^I \mid \alpha^I \in [\underline{\alpha}^I, \overline{\alpha}^I]\} \tag{3.43}$$

and the set $\hat{\Omega}$ and function $\hat{S}_W(z)$ as

$$\hat{\Omega} := \{\tau\beta e^{-\tau\alpha} \mid \alpha \in \partial^r\hat{\Omega}^\alpha, \beta \in \hat{\Omega}^\beta\}, \tag{3.44}$$

$$\hat{S}_W(z, \alpha^R) := \frac{1}{\tau}\mathrm{Re}[\,W_0(z)\,] + \alpha^R. \tag{3.45}$$

The system (3.2) with the uncertainties (3.32) is robustly stable for the fixed time-delay $\tau > 0$ if and only if

$$\max_{z\in\partial\hat{\Omega}} \hat{S}_W(z, \overline{\alpha}^R) < 0. \tag{3.46}$$

Proof For all β, $S_W(\alpha, \beta, \tau)$ is maximized in $\alpha \in \partial^r\hat{\Omega}^\alpha$ by Lemma 3.1. Thus,

$$\max_{\alpha\in\Omega^\alpha,\beta\in\Omega^\beta} S_W(\alpha, \beta, \tau) = \max_{\alpha\in\partial^r\hat{\Omega}^\alpha,\beta\in\Omega^\beta} S_W(\alpha, \beta, \tau) \tag{3.47}$$

holds. Note that

$$\{S_W(\alpha,\beta,\tau) \mid \alpha \in \partial^r\hat{\Omega}^\alpha, \beta \in \hat{\Omega}^\beta\} = \{\hat{S}_W(z,\overline{\alpha}^R) \mid z \in \hat{\Omega}\}. \tag{3.48}$$

Because $\hat{\Omega}$ is a bounded closed set, $\mathrm{Re}[\,W_0(z)\,]$ for $z \in \hat{\Omega}$ is maximized in $\partial\hat{\Omega}$ due to Lemma 3.8. Hence,

$$\max_{z\in\hat{\Omega}} \hat{S}_W(z,\overline{\alpha}^R) = \max_{z\in\partial\hat{\Omega}} \hat{S}_W(z,\overline{\alpha}^R) \tag{3.49}$$

follows. Consequently,

$$\max_{\alpha\in\Omega^\alpha,\beta\in\Omega^\beta} S_W(\alpha,\beta,\tau) = \max_{z\in\partial\hat{\Omega}} \hat{S}_W(z,\overline{\alpha}^R) \tag{3.50}$$

is realized and then the theorem is proven by Lemma 2.6.

Remark 3.2 It naturally follows that with any $\hat{\hat{\Omega}}$ such that $\partial\hat{\Omega} \subseteq \hat{\hat{\Omega}} \subseteq \hat{\Omega}$, the condition (3.46) can be rewritten to $\forall z \in \hat{\hat{\Omega}}$, $\hat{S}_W(z,\overline{\alpha}^R) < 0$. Note that for $\hat{\Omega}^\alpha$ the only rightmost edge influences on the robust stability.

Remark 3.3 Thanks to the assumption about the form of $\hat{\Omega}^\alpha$ and $\hat{\Omega}^\beta$, Lemma 2.6 can be reduced to a one-parametric search problem from two-parametric one by Theorem 3.9. If $\hat{\Omega}^\alpha$ is allowed to be any bounded closed set without restriction on the real part of α, this reduction cannot be fulfilled due to the second term of S_W.

Remark 3.4 If $\hat{\Omega}^\alpha$ and $\hat{\Omega}^\beta$ are given as a box-type and a sector-type uncertainties respectively, Theorem 3.9 can be further reduced to the extreme point result given in Section 3.1.

By the similar way to the previous section, Theorem 3.9 can be directly extended to the multivariable system (3.1) under Assumption 2.1. Suppose that the uncertain parameters α_i and β_i, $i = 1,\cdots,n$ in (2.31) vary within the constraint

$$\alpha_i \in \hat{\Omega}_i^\alpha, \quad \beta_i \in \hat{\Omega}_i^\beta, \quad i = 1,\cdots,n, \tag{3.51}$$

where the definitions of $\hat{\Omega}_i^\alpha$ and $\hat{\Omega}_i^\beta$, $i = 1,\cdots,n$ are similar to $\hat{\Omega}^\alpha$ and $\hat{\Omega}^\beta$ respectively.

Theorem 3.9 can be generalized to the following boundary implication of stability for the system (3.1) with the uncertainties (3.51).

Corollary 3.10 Under Assumption 2.1, given the right edge of $\hat{\Omega}_i^\alpha$, $i = 1,\cdots,n$ as

$$\partial^r\hat{\Omega}_i^\alpha := \{\overline{\alpha}_i^R + j\alpha_i^I \mid \alpha_i^I \in [\underline{\alpha}_i^I, \overline{\alpha}_i^I]\}, \quad i = 1,\cdots,n \tag{3.52}$$

and

$$\hat{\Omega}_i := \{\tau\beta_i e^{-\tau\alpha_i} \mid \alpha_i \in \partial^r\hat{\Omega}_i^\alpha, \beta_i \in \hat{\Omega}_i^\beta\}, \quad i = 1,\cdots,n, \tag{3.53}$$

the linear time-delay system (3.1) whose characteristic quasi-polynomial is (2.31) with the uncertainties prescribed by (3.51) is robustly stable for the fixed time-delay $\tau > 0$ if and only if

$$\max_{z \in \partial \hat{\Omega}_i} \hat{S}_W(z, \overline{\alpha}_i^R) < 0, \quad i = 1, \cdots, n. \tag{3.54}$$

Proof It is obvious from Theorem 3.9 and Lemma 2.7. □

3.3 Illustrative Examples

The robust stability criteria obtained in the previous sections are demonstrated by the following examples. The first example demonstrates the extreme point result and the second one the boundary implication.

Example 3.1 Consider the time-delay system (3.1) with

$$\begin{aligned} A &= \begin{bmatrix} \alpha^R & \alpha^I \\ -\alpha^I & \alpha^R \end{bmatrix}, & \alpha^R \in [-0.08, 0.02],\ \alpha^I \in [-6, -5], \\ B &= \begin{bmatrix} \beta^r \cos \beta^\theta & \beta^r \sin \beta^\theta \\ -\beta^r \sin \beta^\theta & \beta^r \cos \beta^\theta \end{bmatrix}, & \beta^r \in [0.5, 1.5],\ \beta^\theta \in [0.8, 1.5], \end{aligned} \tag{3.55}$$

where α^R, α^I, β^r and β^θ are uncertain parameters. Since A and B are commutative matrices, the Schur theorem [34] can be applied to triangularize A and B simultaneously as

$$\begin{aligned} U^{-1}AU &= \begin{bmatrix} \alpha^R + j\alpha^I & 0 \\ 0 & \alpha^R - j\alpha^I \end{bmatrix}, \\ U^{-1}BU &= \begin{bmatrix} \beta^r e^{j\beta^\theta} & 0 \\ 0 & \beta^r e^{-j\beta^\theta} \end{bmatrix} \end{aligned}$$

by the unitary matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ j\frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then, the characteristic quasi-polynomial is factorized as

$$\begin{aligned} & \det U^{-1} \det[\, sI - A - Be^{-\tau s} \,] \det U \\ = \;& (s - (\alpha^R + j\alpha^I) - \beta^r e^{j\beta^\theta} e^{-\tau s})(s - (\alpha^R - j\alpha^I) - \beta^r e^{-j\beta^\theta} e^{-\tau s}). \end{aligned} \tag{3.56}$$

The uncertainties are rewritten as

$$\begin{aligned} &\alpha \in \Omega^\alpha, \ \beta \in \Omega^\beta, \\ &\Omega^\alpha = \{\alpha^R + j\alpha^I \mid \alpha^R \in [-0.08, 0.02], \, \alpha^I \in [-6, -5]\}, \\ &\Omega^\beta = \{\beta^r e^{j\beta^\theta} \mid \beta^r \in [0.5, 1.5], \, \beta^\theta \in [0.8, 1.5]\}. \end{aligned} \tag{3.57}$$

We are interested in the delay τ that makes the uncertain quasi-polynomial (3.56) and (3.57) robustly stable. Note that for treating the quasi-polynomial (3.56) with respect to the stability, we have only to concentrate on the left factor of (3.56). Therefore, Theorem 3.4 is applied to the uncertain quasi-polynomial

$$s - \alpha - \beta e^{-\tau s} \tag{3.58}$$

with (3.57).

As τ increases from 0, the case specified in Theorem 3.4 popps up as follows.

$$\begin{array}{rcll} 0 < & \tau & < \dfrac{2\pi - 1.5}{6} & : (\mathrm{II}) \\ \dfrac{2i\pi - 1.5}{6} \leq & \tau & \leq \dfrac{2i\pi - 0.8}{5} & : (\mathrm{I}) \quad \text{for} \quad i = 1, 2, 3, 4 \\ \dfrac{2i\pi - 0.8}{5} < & \tau & < \dfrac{2(i+1)\pi - 1.5}{6} & : (\mathrm{II}) \quad \text{for} \quad i = 1, 2, 3, 4 \\ \dfrac{10\pi - 1.5}{6} \leq & \tau & & : (\mathrm{I}) \end{array}$$

In order to check Theorem 3.4, the graphs of (3.25) and (3.26) where τ is a variable are visualized as in Figure 3.11. Such the graphs can be easily composed with the help of the Mathematica command “ProductLog” which calculates any branches of the Lambert W function. Of course, this can be done by Maple and Matlab as well. Figure 3.11 tells that the above system is robustly stable for $0.23 < \tau < 0.42$.

Example 3.2 Let the coefficient matrices of the time-delay system (3.1) be

$$A = \begin{bmatrix} \alpha^R & \alpha^I \\ -\alpha^I & \alpha^R \end{bmatrix}, \quad B = \begin{bmatrix} \beta^R & \beta^I \\ -\beta^I & \beta^R \end{bmatrix}, \tag{3.59}$$

where α^R, α^I, β^R and β^I are real uncertain parameters. Note that A and B are simultaneously triangularisable and the scalar complex-valued system

$$\dot{x}(t) = (\alpha^R + j\alpha^I)x(t) + (\beta^R + j\beta^I)x(t - \tau) \tag{3.60}$$

whose characteristic quasi-polynomial is

$$s - (\alpha^R + j\alpha^I) - (\beta^R + j\beta^I)e^{-\tau s} \tag{3.61}$$

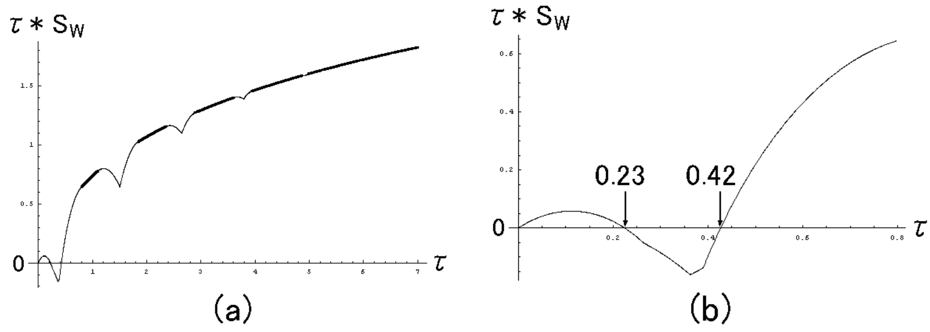

Figure 3.11: (a) The thin line and the thick line represents the graph of (3.26) and (3.25) respectively. (b) Blowup of the stable interval in the graph (a). For visualizability, the graphs are scaled by τ.

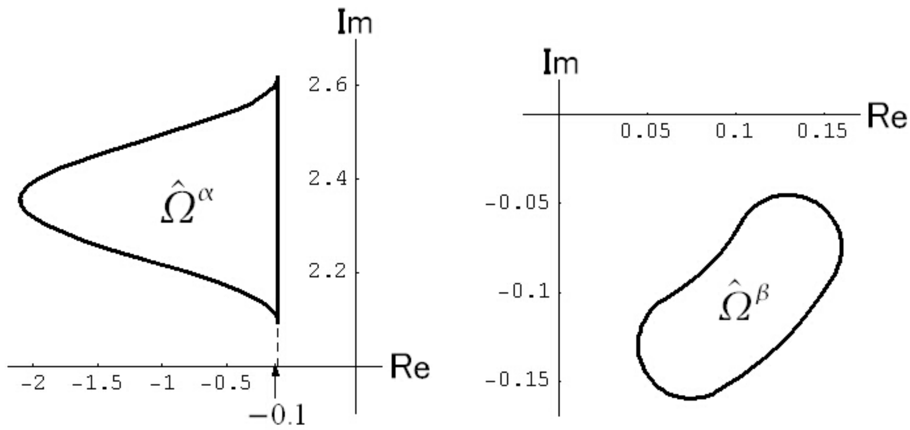

Figure 3.12: $\hat{\Omega}^\alpha$ in (3.62) and $\hat{\Omega}^\beta$ in (3.63).

is equivalent to the system (3.59) in stability likewise to the previous example.

With functions

$$\alpha^R(R, \alpha^I) := R(\cos(12\alpha^I - 8\pi) - 1) - 0.1,$$

$$\beta(r, \theta) := \begin{cases} (0.15 + r)e^{j\left(-\frac{\pi}{6}(\theta - 0) + \frac{\pi}{3}(\theta - 1)\right)}, & \theta \in [0, 1), \\ \left(0.15 + re^{j\pi(\theta - 1)}\right)e^{-j\frac{\pi}{6}}, & \theta \in [1, 2), \\ (0.15 - r)e^{j\left(-\frac{\pi}{3}(\theta - 2) + \frac{\pi}{6}(\theta - 3)\right)}, & \theta \in [2, 3), \\ \left(0.15 + re^{j\pi(\theta - 4)}\right)e^{-j\frac{\pi}{3}}, & \theta \in [3, 4), \end{cases}$$

the uncertainties are supposed to be expressed by

$$\hat{\Omega}^{\alpha} = \left\{\alpha^R(R,\alpha^I) + j\alpha^I \mid R \in [0,1], \alpha^I \in \left[\tfrac{2\pi}{3}, \tfrac{5\pi}{6}\right]\right\}, \quad (3.62)$$

$$\hat{\Omega}^{\beta} = \{\beta(r,\theta) \mid r \in [0, 0.03], \theta \in [0,4)\}. \quad (3.63)$$

The shapes of $\hat{\Omega}^a$ and $\hat{\Omega}^b$ are depicted as in Figure 3.12. Note that both $\hat{\Omega}^a$ and $\hat{\Omega}^b$ are non-convex sets. We are now interested in the delay margin guaranteeing the robust stability of the system (3.60) with the uncertainties (3.62) and (3.63).

We have

$$\hat{\Omega} = \left\{\tau\beta(r,\theta)e^{0.1\tau}e^{-j\tau\alpha^I} \mid \alpha^I \in \left[\tfrac{2\pi}{3}, \tfrac{5\pi}{6}\right], r \in [0, 0.03], \theta \in [0,4)\right\}. \quad (3.64)$$

Set $\underline{\varphi}(\tau) = -\frac{\pi}{3} - \frac{5\pi}{6}\tau$ and $\overline{\varphi}(\tau) = -\frac{\pi}{5} - \frac{2\pi}{3}\tau$ and define

$$z(\tau,\theta) := \begin{cases} (0.15+0.03)e^{j(\overline{\varphi}(\tau)(\theta-0)-\underline{\varphi}(\tau)(\theta-1))}, & \theta \in [0,1), \\ \left(0.15+0.03e^{j\pi(\theta-1)}\right)e^{j\overline{\varphi}(\tau)}, & \theta \in [1,2), \\ (0.15-0.03)e^{j(\underline{\varphi}(\tau)(\theta-2)-\overline{\varphi}(\tau)(\theta-3))}, & \theta \in [2,3), \\ \left(0.15+0.03e^{j\pi(\theta-4)}\right)e^{j\underline{\varphi}(\tau)}, & \theta \in [3,4), \end{cases}$$

$$C_z := \{\tau e^{0.1\tau}z(\tau,\theta) \mid \theta \in [0,4)\}.$$

Then, $\partial\hat{\Omega} = C_z$ or $\partial\hat{\Omega} \subset C_z$ holds depending on the τ values. Figure 3.13 shows the relations between $\partial\hat{\Omega}$ and C_z for $\tau = 5$ and $\tau = 13$. The condition (3.46) of Theorem 3.9 is therefore equivalent to $\forall z \in C_z$, $\hat{S}_W(z, -0.1) < 0$, i.e. $\forall\theta \in [0,4)$, $\hat{S}_W(\tau e^{0.1\tau}z(\tau,\theta), -0.1) < 0$ for the fixed τ (see also Remark 3.2).

In order to obtain the desired delay margin, we should plot the function $\hat{S}_W(\tau e^{0.1\tau}z(\tau,\theta), -0.1)$ versus τ and θ using "Plot3D" function and "ProductLog" function of Mathematica as shown in Figure 3.14. In Figure 3.14, the two intervals of τ such that the surface of the function $\hat{S}_W$ does not mount up the horizontal plane for all $\theta \in [0,4)$ correspond to the delay margins satisfying the condition $\forall\theta \in [0,4)$, $\hat{S}_W(\tau e^{0.1\tau}z(\tau,\theta), -0.1) < 0$, and they are indicated by the two areas between the two pairs of the bold lines: one corresponds to $\tau = 0.29, 1.50$, the other corresponds to $\tau = 3.51, 3.78$. From this figure, it can be verified that the system (3.60), namely the system (3.59), with the uncertainties (3.62) and (3.63) is robustly stable for $0.29 < \tau < 1.50$ or $3.51 < \tau < 3.78$.

3.4 Concluding Remarks

In this chapter, robust stability of linear systems with a single delay have been investigated by means of the Lambert W function. As to this, the extreme point results and the boundary implications of stability were elucidated in Section 3.1 and 3.2 respectively. The extreme

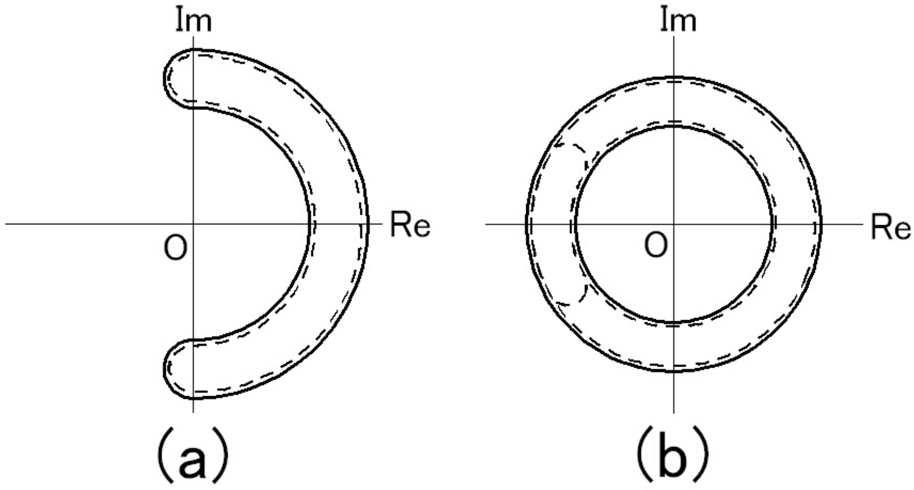

Figure 3.13: The bold line and the dashed line represent $\partial\tilde{\hat{\Omega}}$ and C_z respectively: (a) $\partial\hat{\Omega} = C_z$ holds for $\tau = 5$. (b) $\partial\hat{\Omega} \subset C_z$ holds for $\tau = 13$.

point results reduced a robust stability check of the linear time-delay system (3.1) to a simple test in a few boundary points. The boundary implications, which are generalizations of the extreme point results, converted a two-parametric search problem demanded when checking the robust stability using Lemma 2.6 or 2.7 to a single-parametric one.

The virtue of the Lambert W fucntion is that the critical characteristic root for their stability can be picked out among the all characteristic roots. As a result, it gives us simple and exact stability criteria for a class of linear time-delay systems, more specifically such a class as those that fulfills Assumption 2.1. Furthermore, it can be numerically computed using Mathematica, Maple or Matlab and therefore the extreme point results and boundary implications can be easily checked as demonstrated in Section 3.3. Assumption 2.1 corresponds to a simultaneously triangularizable condition in the state space (see Remark 2.4). This class may be rather restrictive and, as a matter of fact, it is equivalent to a set of scalar systems. The essential reason of making this assumption is that the Lambert W function is defined as a scalar function. In [39, 102], some extensions to the matrix form of the Lambert W function have been attempted. They were at the inchoate stage and therefore have room for improvement.

The Lambert W function approach can be applied to only single time-delay systems such as (3.1), so it is not suitable to multiple delay systems. In the subsequent chapter, commensurate time-delay systems are attempted to cope with by a control technique. Meanwhile, the restriction as to the simultaneous triangularizability can be overcome in

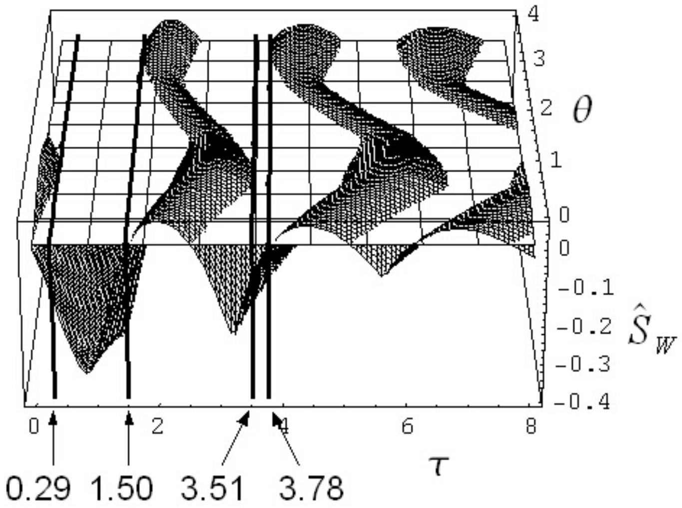

Figure 3.14: $\hat{S}_W(\tau e^{0.1\tau} z(\tau, \theta), -0.1)$ against $\tau \in (0, 8)$ and $\theta \in [0, 4)$.

some sense.

This chapter is concluded with the remark that the progress of the Lambert W function approach for the linear time-delay systems are still underway.

Chapter 4

Stabilization Strategy

In this chapter, a new stabilization strategy is developed in the framework of the Lambert W function approach. As mentioned in the previous chapters, for making use of the Lambert W function, Assumption 2.1 has to be put into target systems. In order to relax this restriction, one may consider that some structural changes are made to the systems by feedback controllers. In this chapter, *decoupling control* [87] is adopted for this purpose, which reorganizes the target systems into more suitable forms for the Lambert W function approach. Note that for the complete forms so as to fit the Lambert W function, further restrictions might be imposed to the decoupled systems.

In Section 4.1, a decoupling technique of [87] for linear commensurate time-delay systems are reviewed. The paper [87] only addresses the decoupling technique but does not mention stabilization problems. For the sake of stabilization of decoupled systems, a novel pole placement technique taking advantage of the Lambert W function is proposed in Section 4.2. Combining this technique with the decoupling control, a new stabilization scheme is given in Section 4.3. This scheme is examined in Section 4.4 by numerical examples. Section 4.5 gives further comments concerning the topics of this chapter.

4.1 Decoupling Control

Generally speaking, systems are said to be *decoupled* if the input-output dependency is altered into one-to-one dependency by feedback controllers. In [87], a decoupling control method for linear time-delay systems is given in terms of matrix forms. This section outlines this control scheme.

Consider a commensurate linear time-delay system with m-inputs and m-outputs

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{N_a} A_i x(t-i\tau) + \sum_{i=0}^{N_b} B_i u(t-i\tau), \\ y(t) &= \sum_{i=0}^{N_c} C_i x(t-i\tau), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbf{R}^n$ denotes states, $u(t) \in \mathbf{R}^m$ inputs, $y(t) \in \mathbf{R}^m$ outputs, $\tau > 0$ a constant time-delay, $A_i \in \mathbf{R}^{n\times n}$, $i = 0, 1, \cdots, N_a$, $B_i \in \mathbf{R}^{n\times m}$, $i = 0, 1, \cdots, N_b$, $C_i \in \mathbf{R}^{m\times n}$, $i = 0, 1, \cdots, N_c$ and N_a, N_b and N_c are non-negative integers. Define a delay operator Δ as in (2.9) and give matrix representations

$$A(\Delta) := \sum_{i=0}^{N_a} A_i \Delta^i, \quad B(\Delta) := \sum_{i=0}^{N_b} B_i \Delta^i, \quad C(\Delta) := \sum_{i=0}^{N_c} C_i \Delta^i. \tag{4.2}$$

Then, the time-delay system (4.1) is redefined as

$$\begin{aligned} \dot{x}(t) &= A(\Delta)x(t) + B(\Delta)u(t), \\ y(t) &= C(\Delta)x(t). \end{aligned} \tag{4.3}$$

Note that the elements of $A(\Delta)$, $B(\Delta)$ and $C(\Delta)$ are all classified into a real polynomial ring of Δ denoted by $\mathbf{R}[\Delta]$.

Taking Laplace transform of the system (4.3), the transfer function matrix

$$G_o(s, \sigma) = C(\sigma)(sI - A(\sigma))^{-1}B(\sigma), \tag{4.4}$$

with $\sigma = e^{-\tau s}$ as in (2.14), is obtained. Here, it is noticed that a polynomial matrix $M(\Delta)$ bears the same structure to the frequency domain, that is formally $\mathcal{L}[\,M(\Delta)\,] = M(\sigma)$. This is also true even for rational function matrices of Δ.

The first step to decouple the system (4.3) is to find out non-negative integers n_i, $i = 1, \cdots, m$ according to the procedure below: for each $i = 1, \cdots, m$, substitute, if any, the minimum positive integer $k \le n$ such that $c_i(\sigma)A^{k-1}(\sigma)B(\sigma) \not\equiv 0$, where $c_i(\sigma)$ is the ith row of $C(\sigma)$ and ($\equiv$) means "identically equal", into n_i, if not, let $n_i = n$. The aim in the decoupling scheme of [87] is to achieve input-output in dependency

$$y_i^{(n_i)}(t) + \sum_{k=0}^{n_i-1} \mu_{ik}(\Delta) y_i^{(k)}(t) = \lambda_i(\Delta)v_i(t), \quad i = 1, \cdots, m \tag{4.5}$$

by feedback control law

$$u(t) = F(\Delta)x(t) + G(\Delta)v(t), \tag{4.6}$$

where $v(t) \in \mathbf{R}^m$ is external inputs, $y_i(t)$ and $v_i(t)$ represent the ith element of $y(t)$ and $v(t)$ respectively, $\mu_{ik}(\Delta) \in \mathbf{R}[\Delta]$, $k = 0, \cdots, n_i-1$, $i = 1, \cdots, m$, $\lambda_i(\Delta) \in \mathbf{R}[\Delta]$, $i = 1, \cdots, m$,

$F(\Delta) \in \mathbf{R}^{m\times n}(\Delta)$ and $G(\Delta) \in \mathbf{R}^{m\times m}(\Delta)$ in which $\mathbf{R}(\Delta)$ denotes real rational functions of Δ. The diagonal transfer function matrix of the system (4.5) is

$$G_c(s,\sigma) = \mathrm{diag}\left[\frac{\lambda_1(\sigma)}{s^{n_1} + \sum_{k=0}^{n_1-1} \mu_{1k}(\sigma)s^k}\ ,\ \cdots\ ,\ \frac{\lambda_m(\sigma)}{s^{n_m} + \sum_{k=0}^{n_m-1} \mu_{mk}(\sigma)s^k}\right]. \tag{4.7}$$

Here, n_i, $i = 1,\cdots,m$ stand for the number of integrators included in the ith path of the decoupled system (4.5).

The paper [87] targets its own goal such that the feedback (4.6) is realized by *non-predictive*, *stable* and *regular* rules: they are defined as follows.

Definition 4.1 A rational function matrix $M(\sigma)$ is *non-predictive* if all the denominators of the elements have a nonzero constant. The feedback law (4.6) is *non-predictive* if both $F(\sigma)$ and $G(\sigma)$ are non-predictive.

Definition 4.2 A rational function matrix $M(\sigma)$ is *stable* if for all the denominators of the elements the moduli of the roots are all greater than one[1]. The feedback law (4.6) is *stable* if both $F(\sigma)$ and $G(\sigma)$ are stable.

Definition 4.3 The feedback law (4.6) is *regular* if $G(\sigma)$ is nonsingular and $G^{-1}(\sigma)$ is non-predictive[2].

Based on the above system construction, define some parameters providing decoupla-

[1]The roots of a polynomial $p(e^{-\tau s})$ all lie in the complex open left half-plane if and only if the moduli of the roots of $p(\sigma)$ are all greater than one.

[2]In [87], although the regularity is defined by adding stability of $G^{-1}(\sigma)$ to the above conditions, stability of $G^{-1}(\sigma)$ does not seems be claimed in the feedback law even though we cite here their statement.

bility by non-predictive, stable and regular feedback as below:

$$\hat{D}(\sigma) := \begin{bmatrix} c_1(\sigma)A^{n_1-1}(\sigma)B(\sigma) \\ \vdots \\ c_m(\sigma)A^{n_m-1}(\sigma)B(\sigma) \end{bmatrix}, \tag{4.8}$$

$$\hat{E}(\sigma) := \begin{bmatrix} c_1(\sigma)A^{n_1}(\sigma) \\ \vdots \\ c_m(\sigma)A^{n_m}(\sigma) \end{bmatrix}, \tag{4.9}$$

$$\hat{\hat{E}}(\sigma) := \begin{bmatrix} \sum_{k=0}^{n_1-1} \mu_{1k}(\sigma)c_1(\sigma)A^k(\sigma) \\ \vdots \\ \sum_{k=0}^{n_m-1} \mu_{mk}(\sigma)c_m(\sigma)A^k(\sigma) \end{bmatrix}, \tag{4.10}$$

$$\Lambda(\sigma) := \mathrm{diag}\,[\,\lambda_1(\sigma), \cdots, \lambda_m(\sigma)\,], \tag{4.11}$$

$$\nu_i := \min\left\{ k \in \mathbf{Z}^+ \;\middle|\; \lim_{z\to 0} \frac{1}{z^k} c_i(z)A^{n_i-1}(z)B(z) \neq 0 \right\}, \quad i = 1, \cdots, m, \tag{4.12}$$

$$\hat{\hat{D}}(\sigma) := \begin{bmatrix} \frac{1}{\sigma^{\nu_i}} c_1(\sigma)A^{n_1-1}(\sigma)B(\sigma) \\ \vdots \\ \frac{1}{\sigma^{\nu_m}} c_m(\sigma)A^{n_m-1}(\sigma)B(\sigma) \end{bmatrix}, \tag{4.13}$$

$$E(\sigma) := \hat{E}(\sigma) + \hat{\hat{E}}(\sigma), \tag{4.14}$$

$$a_i := \min\left\{ k \in \mathbf{Z}^+ \;\middle|\; \lim_{z\to 0} z^k \left[\left(\hat{D}^{-1}(z) \right)^T \right]_i < \infty \right\}, \quad i = 1, \cdots, m, \tag{4.15}$$

$$b_i := \max\left\{ k \in \mathbf{Z}^+ \;\middle|\; \lim_{z\to 0} \frac{1}{z^k} e_i(z) < \infty \right\}, \quad i = 1, \cdots, m, \tag{4.16}$$

$$\Gamma := \{\sigma \neq 0 \mid \det \hat{D}(\sigma) = 0\}, \tag{4.17}$$

where $\mathbf{Z}^+$ represents non-negative integer set, $[\,(\hat{D}^{-1}(z))^T\,]_i$ is the ith row of the matrix $(\hat{D}^{-1}(z))^T$ and $e_i(z)$ the ith row of $E(\sigma)$. Notice that $\hat{\hat{D}}(\sigma) \in \mathbf{R}^{m\times m}[\sigma]$.

Then, Theorem 4.4 below gives a condition for the existence of non-predictive, stable and regular feedback law (4.6) such that the system (4.3) is decoupled to (4.5) [87].

Theorem 4.4 The linear time-delay system (4.3) can be decoupled to the system (4.5) with non-predictive, stable and regular feedback (4.6) if and only if the following conditions are all satisfied:

(I) Letting $\hat{\hat{D}}_0$ be the constant term of $\hat{\hat{D}}(\sigma)$, $\hat{\hat{D}}_0$ is nonsingular.

(II) $\forall \sigma \in \Gamma$, $|\sigma| > 1$ or $\Gamma = \emptyset$.

(III) $b_i \geq a_i$, $i = 1, \cdots, m$ hold for given $\mu_{ik}(\sigma)$, $k = 0, 1, \cdots, n_i - 1$, $i = 1, \cdots, m$.

Chosen $\Lambda(\sigma)$ as

$$\lambda_i(\sigma) = \sigma^{\nu_i} w_i(\sigma), \quad i = 1, \cdots, m, \tag{4.18}$$

where $w_i(\sigma)$, $i = 1, \cdots, m$ are arbitrary real polynomials of σ with nonzero constant terms, the feedback law

$$F(\sigma) = -\hat{D}^{-1}(\sigma)E(\sigma) \tag{4.19}$$

$$G(\sigma) = \hat{D}^{-1}(\sigma)\Lambda(\sigma), \tag{4.20}$$

can transform the system (4.3) into the decoupled form as follows:

$$\begin{aligned} & C(\sigma)\left(sI - A(\sigma) - B(\sigma)F(\sigma)\right)^{-1} B(\sigma)G(\sigma) \\ = \ & \mathrm{diag}\left[\frac{\lambda_1(\sigma)}{s^{n_1} + \sum_{k=0}^{n_1-1} \mu_{1k}(\sigma)s^k}, \ \cdots, \ \frac{\lambda_m(\sigma)}{s^{n_m} + \sum_{k=0}^{n_m-1} \mu_{mk}(\sigma)s^k}\right]. \end{aligned} \tag{4.21}$$

Remark 4.1 In Theorem 4.4, the condition (I) guarantees the existence of a decoupling feedback law with regular $G(\sigma)$, (II) the stability of the feedback and (III) the non-predictiveness of it. While (I) and (II) are ruled by the given system construction, one can handle (III) with the design parameters $\mu_{ik}(\sigma)$'s.

4.2 Pole Assignment by the Lambert W Function

In this section, a new pole placement technique for a quasi-polynomial

$$s - \alpha - \beta e^{-\tau s}, \tag{4.22}$$

where $\alpha, \beta \in \mathbf{C}$, is proposed by making use of the Lambert W function. More specifically, the objective of this section is to obtain the pole locations such that every real parts of the roots of (4.22) are less than an arbitrarily chosen value. Difficulty of this objective is that once one intends to assign a pole $s_0 \in \mathbf{C}^-$ to (4.22), the pair of α and β achieving this assignment exists infinitely and locations of the other poles are ambiguous, possibly in the right half-plane. Therefore, α and β must be assigned under proper conditions and this task cannot be accomplished in a trivial way.

As stated in Subsection 2.5.2, the roots of (4.22) are expressed by the Lambert W function W as

$$s = \frac{1}{\tau} W(\tau\beta e^{-\tau\alpha}) + \alpha. \tag{4.23}$$

Moreover, the root

$$s = \frac{1}{\tau} W_0(\tau \beta e^{-\tau\alpha}) + \alpha \tag{4.24}$$

is always in the rightmost of all the roots by Lemma 2.3. If the desired pole s_0 is embedded into (4.24), then it is ensured that s_0 exists in the right position than the others. Accordingly, the problem is to solve the equation

$$s_0 = \frac{1}{\tau} W_0(\tau \beta e^{-\tau\alpha}) + \alpha \tag{4.25}$$

for α and β.

Now let $z_w = \tau\beta e^{-\tau\alpha}$ be unknown. Assume that α is given a priori. Then, α has to meet

$$\alpha \in W_0^\alpha := \left\{ -\frac{1}{\tau} W_0(z_w) + s_0 \;\middle|\; z_w \in \mathbf{C} \right\} \tag{4.26}$$

so as to fulfill (4.25). The region W_0^α is as depicted in Figure 4.1. For a given $\alpha \in W_0^\alpha$, if β satisfies

$$\beta = (s_0 - \alpha) e^{\tau s_0}, \tag{4.27}$$

then it is ensured that s_0 is one of the roots of (4.22) and lies in the rightmost of all the roots.

Conversely, consider the situation where one selects β a priori. Convert (4.25) to

$$\alpha = s_0 - \frac{1}{\tau} W_0(\tau\beta e^{-\tau\alpha}). \tag{4.28}$$

Substituting (4.28) into (4.22) and assuming (4.22) is vanished, we have

$$\begin{aligned} & \frac{1}{\tau} W_0(\tau\beta e^{-\tau\alpha}) - \beta e^{-\tau s_0} = 0 \\ \Leftrightarrow\;\; & W_0(\tau\beta e^{-\tau\alpha}) = \tau\beta e^{-\tau s_0} && (4.29) \\ \Leftrightarrow\;\; & \beta = \frac{e^{\tau s_0}}{\tau} W_0(\tau\beta e^{-\tau\alpha}). && (4.30) \end{aligned}$$

Thus, β has to obey

$$\beta \in W_0^\beta := \left\{ \frac{e^{\tau s_0}}{\tau} W_0(z_w) \;\middle|\; z_w \in \mathbf{C} \right\}, \tag{4.31}$$

in which W_0^β forms as in Figure 4.1. Meanwhile, from (4.29) and the definition of the Lambert W function (2.17) and (2.18),

$$\begin{aligned} & \tau\beta e^{-\tau s_0} e^{\tau\beta e^{-\tau s_0}} = \tau\beta e^{-\tau\alpha} \\ \Leftrightarrow\;\; & e^{\tau\beta e^{-\tau s_0} - \tau s_0} = e^{-\tau\alpha} \\ \Leftrightarrow\;\; & -\tau\alpha = \tau\beta e^{-\tau s_0} - \tau s_0 + j2\pi k, \quad k \in \mathbf{Z} \\ \Leftrightarrow\;\; & \alpha = -\beta e^{-\tau s_0} + s_0 + j\frac{2\pi k}{\tau}, \quad k \in \mathbf{Z}, && (4.32) \end{aligned}$$

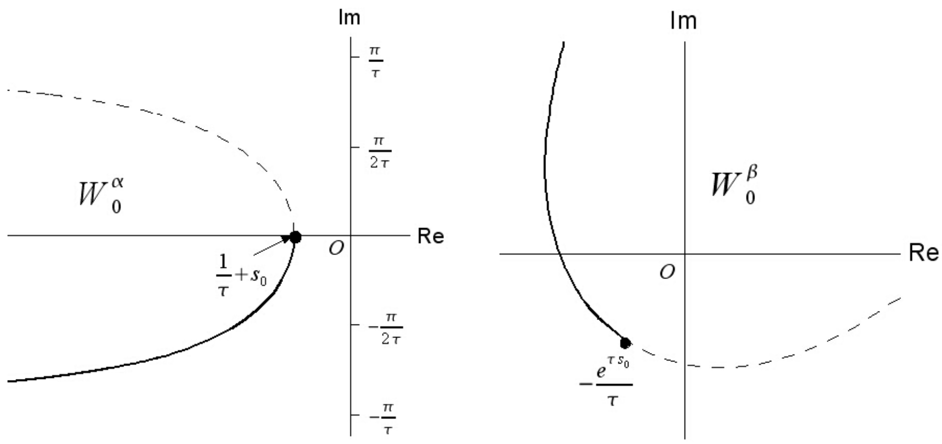

Figure 4.1: Regions of W_0^α and W_0^β: the dashed curves are out of them.

is obtained, where **Z** denotes the integer set. Here, recall the requirement (4.26) for α such that s_0 falls into the principal branch of W. Keeping this in mind, when β is given as in (4.31), $k = 0$ is necessary in (4.32), otherwise α deviates from W_0^α since $-\beta e^{-\tau s_0} + s_0 \in W_0^\alpha$ has to hold and the vertical width of W_0^α is at largest $2\pi/\tau$ taken in the left infinity (see Figure 4.1). As a consequence, for a given $\beta \in W_0^\beta$, α ought to be as

$$\alpha = -\beta e^{-\tau s_0} + s_0. \tag{4.33}$$

The above argument is summarized as the statement below.

Theorem 4.5 For arbitrary $s_0 \in \mathbf{C}$, suppose that α is given as in (4.26) a priori and β as in (4.27) a posteriori, or conversely β is given as in (4.31) a priori and α as (4.33) a posteriori. Then, the quasi-polynomial (4.22) has s_0 as the rightmost root.

Remark 4.2 Intuitively speaking, Theorem 4.5 is such a pole placement technique that makes the rightmost pole s_0 corresponding to the principal branch of W, say a *master pole*, dominate the other poles corresponding to W_k, $k = \pm 1, \cdots, \pm\infty$, say *slave poles*, to follow the left or under (see Remark 2.3) of the master pole. The locations of the slave poles are not arbitrary but depends on the allocations by the branches W_k, $k = \pm 1, \cdots, \pm\infty$. Meanwhile, the real part of the master pole stands for the stability exponent of the obtained pole locations.

Remark 4.3 If there are some constraints on α or β, arbitrary assignment of s_0 might not be achieved; for example, if α is restricted to zero, then the real part of s_0 must be greater than $-1/\tau$ as inferred from Figure 4.1.

Remark 4.4 If one chooses as $\beta = 0$, the obtained pole locations are equivalent to delay-free ones, that is (4.22) only has a pole α. The other poles of (4.22) diverge to the left infinity likewise to finite spectrum assignment [63], indeed $s = W_k(0)/\tau + \alpha \to -\infty$, $k = \pm 1, \pm 2, \cdots, \pm\infty$.

4.3 Stabilization by the Lambert W Function and Decoupling Control

Combining the pole placement technique by the Lambert W function proposed in the previous section with the decoupling control, a new control starategy is given in this section.

To begin with, simply multiplying the denominators of (4.7) results in

$$\left(s^{n_1} + \sum_{k=0}^{n_1-1} \mu_{1k}(\sigma)s^k\right)\cdots\left(s^{n_m} + \sum_{k=0}^{n_m-1} \mu_{mk}(\sigma)s^k\right). \tag{4.34}$$

Then, (4.34) may correspond to the characteristic quasi-polynomial of the closed-loop system $\det[\,sI - A(\sigma) - B(\sigma)F(\sigma)\,]$. If not, one should be cautious about the existence of the hidden modes, especially unstable ones.

Let us factorize (4.34) into the suitable form for the Lambert W function approach:

$$\begin{aligned} s^{n_1} + \sum_{k=0}^{n_1-1} \mu_{1k}(\sigma)s^k &= (s - \alpha_{10} - \beta_{10}\sigma)\cdots(s - \alpha_{1,n_1-1} - \beta_{1,n_1-1}\sigma), \\ &\vdots \\ s^{n_m} + \sum_{k=0}^{n_m-1} \mu_{mk}(\sigma)s^k &= (s - \alpha_{m0} - \beta_{m0}\sigma)\cdots(s - \alpha_{m,n_m-1} - \beta_{m,n_m-1}\sigma). \end{aligned} \tag{4.35}$$

As expected, stability of (4.34) is governed by (4.35). Remind that $\mu_{ik}(\sigma)$, $k = 0, 1, \cdots, n_i-1$, $i = 1, \cdots, m$ are design parameters, that is one can handle the characteristics, especially stability, of (4.34) with selection of α_{ik} and β_{ik}, $k = 0, 1, \cdots, n_i-1$, $i = 1, \cdots, m$.

Summing up, a stabilization procedure by the Lambert W function and the decoupling control is summarized as the following algorithm.

Algorithm 4.1

step1: Compute n_i, $\hat{D}(\sigma)$, ν_i, $\hat{\hat{D}}(\sigma)$ for $i = 1, \cdots, m$ and Γ. Then check whether both (I) and (II) of Theorem 4.4 are true or not. If positive, go to the next step. If negative, stabilization by this approach is failed.

step2: Let

$$s^{n_i} + \sum_{k=0}^{n_i-1} \mu_{ik}(\sigma)s^k = (s - \alpha_{i0} - \beta_{i0}\sigma)\cdots(s - \alpha_{i,n_i-1} - \beta_{i,n_i-1}\sigma), \quad i = 1, \cdots, m. \tag{4.36}$$

Then for each $i = 1, \cdots, m$, expand the right-hand side of (4.36) and substitute the coefficients of s^k, $k = 0, 1, \cdots, n_i - 1$ into $\mu_{ik}(\sigma)$, $k = 0, 1, \cdots, n_i - 1$ respectively.

step3: Compute $\hat{E}(\sigma)$ and a_i, $i = 1, \cdots, m$. Estimating $\hat{\hat{E}}(\sigma)$ and $E(\sigma)$, for each $i = 1, \cdots, m$ constraints imposed on α_{ik} and β_{ik}, $k = 0, 1, \cdots, n_i - 1$ are clarified by the condition (III) of Theorem 4.4. For each $i = 1, \cdots, m$, $b_i \geq a_i$ is valid if and only if the terms of $e_i(\sigma)$ with orders of σ, if any, being less than a_i are all vanished.

step4: Based on the constraints on α_{ik}'s and β_{ik}'s, choose the master poles $s_{ik} \in \mathbf{C}^-$, $k = 0, 1, \cdots, n_i - 1$, $i = 1, \cdots, m$ assigned to $(s - \alpha_{ik} - \beta_{ik}\sigma)$, $k = 0, 1, \cdots, n_i - 1$, $i = 1, \cdots, m$ respectively, and then find out α_{ik} and β_{ik}, $k = 0, 1, \cdots, n_i - 1$, $i = 1, \cdots, m$ following Theorem 4.5. If there is a pole whose real part cannot be negative, stabilization is failed.

step5: Decide $\Lambda(\sigma)$ accoding to (4.18) and again compute $\hat{\hat{E}}(\sigma)$ and $E(\sigma)$ using α_{ik}'s and β_{ik}'s obtained in the previous step. Now the feedback law (4.19) and (4.20) can be in hand. Check whether

$$\det[\, sI - A(\sigma) - B(\sigma)F(\sigma)\,] = \prod_{i=1}^{m} (s - \alpha_{i0} - \beta_{i0}\sigma) \cdots (s - \alpha_{i,n_i-1} - \beta_{i,n_i-1}\sigma) \quad (4.37)$$

holds or not. If positive, stabilization is accomplished. If negative, there are hidden modes estimated by

$$\frac{\det[\, sI - A(\sigma) - B(\sigma)F(\sigma)\,]}{\prod_{i=1}^{m} (s - \alpha_{i0} - \beta_{i0}\sigma) \cdots (s - \alpha_{i,n_i-1} - \beta_{i,n_i-1}\sigma)}. \quad (4.38)$$

If the hidden modes are all stable, stabilization is still done. Otherwise, it is failed.

step6: Finish.

Remark 4.5 Algorithm 4.1 might be unsuccessful in the steps 1, 4 and 5 because of undecouplability of the given system, too severe constraints for non-predictiveness of feedback and existence of unstable hidden modes respectively.

Remark 4.6 When there is no hidden mode, stability exponent of the resultant closed-loop system corresponds to the greatest real part among the master poles chosen in the step 4. If there are stable hidden modes, it is possibly determined by the poles of the modes.

4.4 Illustrative Examples

The stabilization algorithm derived in the previous section is demonstrated by the numerical examples. The first example illustrates the straightforward case where any hidden modes do not occur. In the second example, one way to deal with hidden modes is presented.

Example 4.1 Let coefficient matrices and a time-delay in (4.3) be [87]

$$A(\Delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \Delta \\ 1 & 0 & 0 \end{bmatrix}, \ B(\Delta) = \begin{bmatrix} 2+\Delta & \Delta \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \ C(\Delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$\tau = 0.5.$$

Note that this system is unstable. In this example, it is aimed to stabilize this system and estimate stability delay margin of the closed-loop system. Algorithm 4.1 gives the following results.

step1: Compute the parameters n_1, n_2, $\hat{D}(\sigma)$, ν_1, ν_2, $\hat{\hat{D}}(\sigma)$ and Γ:

$$\begin{aligned}
&n_1 = 1, \ n_2 = 2, \quad \hat{D}(\sigma) = \begin{bmatrix} 2+\sigma & \sigma \\ 0 & \sigma \end{bmatrix}, \\
&\nu_1 = 0, \ \nu_2 = 1, \quad \hat{\hat{D}}(\sigma) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma & \sigma \\ 0 & 0 \end{bmatrix}, \\
&\Gamma = \{\sigma \neq 0 \mid (2+\sigma)\sigma = 0\} = \{-2\}.
\end{aligned}$$

It is easy to see that (I) and (II) of Theorem 4.4 are both satisfied.

step2: Letting

$$\begin{aligned}
&s + \mu_{10}(\sigma) = s - \alpha_{10} - \beta_{10}\sigma, \\
&s^2 + \mu_{21}(\sigma)s + \mu_{20}(\sigma) = (s - \alpha_{20} - \beta_{20}\sigma)(s - \alpha_{21} - \beta_{21}\sigma),
\end{aligned}$$

$\mu_{10}(\sigma)$, $\mu_{20}(\sigma)$ and $\mu_{21}(\sigma)$ are set as

$$\begin{aligned}
\mu_{10}(\sigma) &= -\alpha_{10} - \beta_{10}\sigma, \\
\mu_{20}(\sigma) &= \alpha_{20}\alpha_{21} + (\alpha_{20}\beta_{21} + \alpha_{21}\beta_{20})\sigma, +\beta_{20}\beta_{21}\sigma^2 \\
\mu_{21}(\sigma) &= -\alpha_{20} - \alpha_{21} + (-\beta_{20} - \beta_{21})\sigma.
\end{aligned}$$

step3: $\hat{E}(\sigma)$ and a_1, a_2 are obtained as

$$\hat{E}(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ \sigma & 1 & \sigma \end{bmatrix}, \quad a_1 = 0, \ a_2 = 1.$$

Since $a_1 = 0$, α_{10} and β_{10} can be chosen freely. From the estimations of $\hat{\hat{E}}(\sigma)$ and $E(\sigma)$ and the fact $a_2 = 1$, the σ^0 terms of $e_2(\sigma)$, which equal to

$$e_2(\sigma) = \begin{bmatrix} \alpha_{20}\alpha_{21} & 1 - \alpha_{20} - \alpha_{21} + \alpha_{20}\alpha_{21} & \alpha_{20}\alpha_{21} \end{bmatrix},$$

must be all zero. A pair of

$$\alpha_{20} = 1, \quad \alpha_{21} = 0$$

is one of the achievable choices. The above setup corroborates the assertion that $b_1 = a_1 = 0$ and $b_2 = a_2 = 1$, and thus (III) of Theorem 4.4 is fulfilled.

step4: In this step, master poles, say s_{10}, s_{20} and s_{21}, are assigned to $(s - \alpha_{10} - \beta_{10}\sigma)$, $(s-\alpha_{20}-\beta_{20}\sigma)$ and $(s-\alpha_{21}-\beta_{21}\sigma)$ respectively. Since α_{10} is free, s_{10} can be elected arbitrarily, so let

$$s_{10} = -5.$$

If one chooses as

$$\alpha_{10} = -10$$

a priori, then

$$\beta_{10} = 5e^{-5}$$

automatically follows by Theorem 4.5. Note that $\alpha_{20} = 1$ and $\alpha_{21} = 0$ have been already given in the previous step. Therefore, choices of s_{20} and s_{21} are bounded by these constraints. If

$$s_{20} = -0.8, \quad s_{21} = -2$$

are given, the constraints are satisfied as conjectured by the ranges of W_0^α. Subsequently,

$$\beta_{20} = -1.8e^{-0.4}, \quad \beta_{21} = -2e^{-1}$$

are again automatic by Theorem 4.5.

step5: Let

$$\Lambda(\sigma) = \mathrm{diag}[\,1,\ \sigma\,].$$

$\hat{\hat{E}}(\sigma)$ and $E(\sigma)$ are recalculated as in the following:

$$\hat{\hat{E}}(\sigma) = \begin{bmatrix} 10-5e^{-5}\sigma & 0 & 0 \\ 0 & -1+1.8e^{-0.4}\sigma+3.6e^{-1.4}\sigma^2 & -\sigma+\left(2e^{-1}+1.8e^{-0.4}\right)\sigma^2 \end{bmatrix},$$

$$E(\sigma) = \begin{bmatrix} 11 - 5e^{-5}\sigma & 0 & 0 \\ \sigma & 1.8e^{-0.4}\sigma + 3.6e^{-1.4}\sigma^2 & \left(2e^{-1} + 1.8e^{-0.4}\right)\sigma^2 \end{bmatrix}.$$

Thus the feedback law (4.19) and (4.20) can be obtained as

$$F(\sigma) = \begin{bmatrix} \dfrac{e^5(-11+\sigma)+5\sigma}{e^5(2+\sigma)} & \dfrac{9\sigma(e+2\sigma)}{5e^{1.4}(2+\sigma)} & \dfrac{(10+9e^{0.6})\sigma^2}{5e(2+\sigma)} \\ -1 & -\dfrac{9(e+2\sigma)}{5e^{1.4}} & -\dfrac{(10+9e^{0.6})\sigma}{5e} \end{bmatrix},$$

$$G(\sigma) = \begin{bmatrix} \dfrac{1}{2+\sigma} & -\dfrac{\sigma}{2+\sigma} \\ 0 & 1 \end{bmatrix}.$$

Furthermore, since

$$\det[\,sI - A(\sigma) - B(\sigma)F(\sigma)\,] = (s-\alpha_{10}-\beta_{10}\sigma)(s-\alpha_{20}-\beta_{20}\sigma)(s-\alpha_{21}-\beta_{21}\sigma)$$

takes place, any hidden modes do not emerge. As a result, the decoupled system

$$\begin{aligned} &\dot{y}_1(t) + 10y_1(t) - 5e^{-5}y_1(t-0.5) = v_1(t), \\ &\ddot{y}_2(t) - \dot{y}_2(t) + \frac{9e^{-0.4}+10e^{-1}}{5}\dot{y}_2(t-0.5) \\ &\qquad -2e^{-1}y_2(t-0.5) + \frac{18e^{-1.4}-10e^{-1}}{5}y_2(t-1) = v_2(t-0.5), \end{aligned}$$

whose transfer function matrix is

$$\mathrm{diag}\left[\ \frac{1}{s+10-5e^{-5}\sigma},\ \ \frac{\sigma}{(s-1+1.8e^{-0.4}\sigma)\,(s+2e^{-1}\sigma)}\ \right],$$

is stabilized.

step6: Finish.

Finally, let us estimate stability delay margin of the decoupled system. For this, one should compose graphs of $S_W(\alpha_{10},\beta_{10},\tau)$, $S_W(\alpha_{20},\beta_{20},\tau)$ and $S_W(\alpha_{21},\beta_{21},\tau)$ with respect

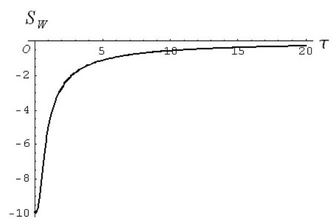

Figure 4.2: $S_W(\alpha_{10},\beta_{10},\tau)$ against τ.

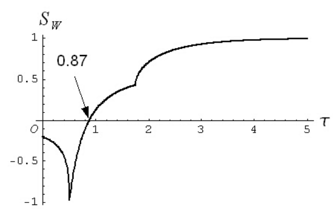

Figure 4.3: $S_W(\alpha_{20},\beta_{20},\tau)$ against τ.

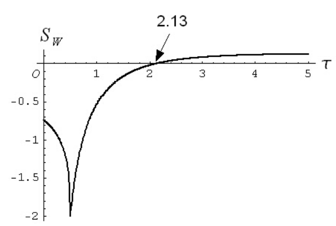

Figure 4.4: $S_W(\alpha_{21},\beta_{21},\tau)$ against τ.

to τ. Figures 4.2–4.4 drawn with the help of Mathematica show that the decoupled system is stable for $0 < \tau < 0.87$. The occurances of $S_W(\alpha_{10}, \beta_{10}, \tau) = -5$, $S_W(\alpha_{20}, \beta_{20}, \tau) = -0.8$ and $S_W(\alpha_{21}, \beta_{21}, \tau) = -2$ can be confirmed at $\tau = 0.5$ in these graphs. It should be noted that one can carry out this task with Maple or Matlab.

Example 4.2 Consider the time-delay system (4.3) with [94]

$$A(\Delta) = \begin{bmatrix} \Delta & 1+2\Delta & \Delta \\ 2+2\Delta & 3 & 3\Delta \\ 1+2\Delta & 1+\Delta & 1+\Delta \end{bmatrix}, \ B(\Delta) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C(\Delta) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\tau = 1.$$

This system is unstable, therefore first it is stabilized and then stability delay margin of the closed-loop system are estimated by the same way to the previous example. Algorithm 4.1 leads to the following results.

step1: Compute the parameters n_1, n_2, $\hat{D}(\sigma)$, ν_1, ν_2, $\hat{\hat{D}}(\sigma)$ and Γ:

$$n_1 = 1,\ n_2 = 1, \quad \hat{D}(\sigma) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\nu_1 = 0,\ \nu_2 = 0, \quad \hat{\hat{D}}(\sigma) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\Gamma = \emptyset.$$

So (I) and (II) of Theorem 4.4 are satisfied.

step2: For given

$$s + \mu_{10}(\sigma) = s - \alpha_{10} - \beta_{10}\sigma,$$
$$s + \mu_{20}(\sigma) = s - \alpha_{20} - \beta_{20}\sigma,$$

we have

$$\mu_{10}(\sigma) = -\alpha_{10} - \beta_{10}\sigma,$$
$$\mu_{20}(\sigma) = -\alpha_{20} - \beta_{20}\sigma.$$

step3: $\hat{E}(\sigma)$ and a_1, a_2 are obtained as

$$\hat{E}(\sigma) = \begin{bmatrix} 2+3\sigma & 4+2\sigma & 4\sigma \\ 1+2\sigma & 1+\sigma & 1+\sigma \end{bmatrix}, \quad a_1 = 0,\ a_2 = 0.$$

Hence there is no constraint on α_{10}, β_{10}, α_{20} and β_{20}, i.e. $b_i \geq a_i$, $i = 1, 2$ hold true for any selection of α_{10}, β_{10}, α_{20} and β_{20}.

step4: Master poles s_{10} and s_{20} assigned to $(s-\alpha_{10}-\beta_{10}\sigma)$ and $(s-\alpha_{20}-\beta_{20}\sigma)$ respectively can be freely chosen. Now we set

$$s_{10} = -10, \quad s_{20} = -10,$$
$$\alpha_{10} = -20, \quad \alpha_{20} = -20,$$

Then Theorem 4.5 gives

$$\beta_{10} = 10e^{-10}, \quad \beta_{20} = 10e^{-10}.$$

step5: Let

$$\Lambda(\sigma) = \mathrm{diag}[\,1,\ 1\,],$$

and $\hat{\hat{E}}(\sigma)$ and $E(\sigma)$ result in

$$\hat{\hat{E}}(\sigma) = \begin{bmatrix} 20-10e^{-10}\sigma & 20-10e^{-10}\sigma & 0 \\ 0 & 0 & 20-10e^{-10}\sigma \end{bmatrix},$$
$$E(\sigma) = \begin{bmatrix} 22+(3-10e^{-10})\sigma & 24+(2-10e^{-10})\sigma & 4\sigma \\ 1+2\sigma & 1+\sigma & 21+(1-10e^{-10})\sigma \end{bmatrix}.$$

Consequently, the feedback law (4.19) and (4.20) is given as

$$F(\sigma) = -E(\sigma)$$
$$G(\sigma) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here, we can see that

$$\det[\,sI - A(\sigma) - B(\sigma)F(\sigma)\,] \neq (s-\alpha_{10}-\beta_{10}\sigma)(s-\alpha_{20}-\beta_{20}\sigma),$$

so that the closed-loop system involves some hidden modes. They show up by the evaluation

$$\frac{\det[\,sI - A(\sigma) - B(\sigma)F(\sigma)\,]}{(s-\alpha_{10}-\beta_{10}\sigma)(s-\alpha_{20}-\beta_{20}\sigma)} = s+1+\sigma.$$

Stablity of the hidden mode $s+1+\sigma$ can be readily tested by the Lambert W function. Indeed the function S_W yields

$$S_W(-1,-1,1) = -0.605 < 0.$$

Following Lemma 2.4, it is confirmd that the hidden mode $s+1+\sigma$ is stable, and thus the resultant decoupled system

$$\dot{y}_1(t) + 20y_1(t) - 10e^{-10}y_1(t-1) = v_1(t),$$
$$\dot{y}_2(t) + 20y_2(t) - 10e^{-10}y_2(t-1) = v_2(t),$$

with the transfer function matrix

$$\mathrm{diag}\left[\ \frac{1}{s+20-10e^{-10}\sigma},\ \ \frac{1}{s+20-10e^{-10}\sigma}\ \right],$$

is also stable.

Stability delay margin of the decoupled system can be observed in Figures 4.5 and 4.6, the latter illustrates the hidden mode, indicating that the closed-loop system is stable for any $\tau > 0$.

4.5 Concluding Remarks

This chapter has proposed a new control scheme for the commensurate linear time-delay system (4.3) based on the decoupling control of [87] and the pole placement technique by the Lambert W function. This pole placement technique was developed in Section 4.2 and the guideline was addressed in Theorem 4.5. The procedure for achieving stabilization was sammurized in Algorithm 4.1.

The decoupling control of [87] leads to non-predictive feedback. Therefore, the proposed stabilization method does not suffer from the troubles caused by the predictive controls entailing numerical integrations as mentioned in Section 2.4. However, since the proposed method is founded on the decoupling control of [87], it becomes unavailable if the non-predictive control, which demands the condition (III) of Theorem 4.4, cannot be admitted. In this case, one has to use general rational function controllers resulting in to carry out some predictive control instead of the non-predictive one.

In Example 4.2, emergence of hidden modes in the closed-loop systems became a problem. Fortunately, their stability could be checked by the Lambert W function in this

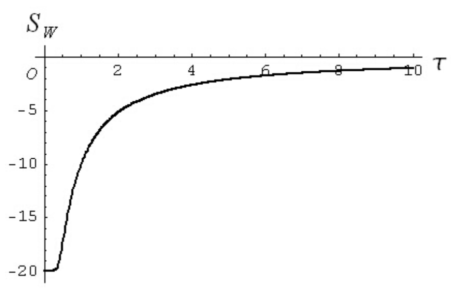

Figure 4.5: $S_W(\alpha_{i0}, \beta_{i0}, \tau)$, $i = 1, 2$ against τ.

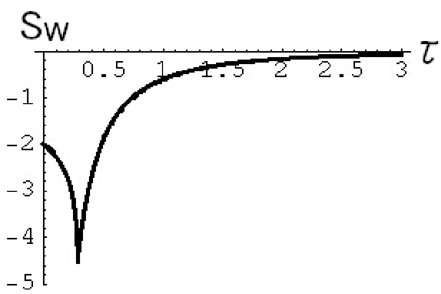

Figure 4.6: $S_W(-1, -1, \tau)$ against τ.

example. However, if this function is not available, one has to use other methods for stability check of hidden modes. To draw up general guidelines for dealing with such modes is left to a future investigation.

The proposed pole placement technique is a generalization of the finite spectrum assignment. An advantage over the finite spectrum assignment is to be easy to compute stability delay margin of the closed-loop systems as demonstrated in Section 4.4. This is due to leaving the delay effect on the closed-loop systems unlike the class of predictive controls. To further bring out the contrast between the proposed method and the finite spectrum assignment is one of interesting issues.

Chapter 5

Additional Dynamics Analysis

This chapter concerns another issue; stability of *additional dynamics* of linear time-delay systems induced by *model transformations*. Similarly to the hereto chapters, the Lambert W function is employed for this purpose.

Model transformations are needed when some kinds of the Lyapunov approaches are adopted [24, 56, 57]. However, such transformations bring about additional dynamics to the transformed systems and as a result conservative stability conditions may be derived. If the additional dynamics is stable, such consevativeness is not involved. This suggests an importance of stability analysis of the additional dynamics. Indeed, this has been intensively done in [24–26, 47, 48, 50], yet this thesis revisits this topic and gives a new insight into the issue taking advantage of the Lambet W function.

This chapter is organized as follows. In Section 5.1, additional dynamics is digested and the underlying problem is exposed. In paticular, first-order and second-order model transformations are dealt with in this thesis. In Section 5.2, stability of additional dynamics is investigated in terms of the Lambert W function and stability and robust stability conditions are given for each of the first-order and second-order transformations. Section 5.3 presents numerical examples and from these examples an inclusion relation between the first-order and second-order transformations is elucidated. Section 5.4 gives a conclusion of this chapter.

5.1 Additional Dynamics

This first section gives the introduction of additional dynamics induced by model transformations of linear time-delay systems based on [25, 26].

Consider a linear single time-delay system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{5.1}$$

where $A, B \in \mathbf{R}^{n\times n}$, $x(t) \in \mathbf{R}^n$ and $\tau > 0$ is a time-delay. In the identity

$$x(t-\tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t+\theta)d\theta, \tag{5.2}$$

replacing $\dot{x}(t+\theta)$ by (5.1), one can obtain

$$x(t-\tau) = x(t) - \int_{-\tau}^{0} [\,Ax(t+\theta) + Bx(t-\tau+\theta)\,]d\theta. \tag{5.3}$$

Substituting (5.3) into (5.1) results in the additional dynamics system such as

$$\dot{x}(t) = (A+B)x(t) - B\int_{-\tau}^{0} [\,Ax(t+\theta) + Bx(t-\tau+\theta)\,]d\theta. \tag{5.4}$$

Note that the system (5.4) is classified as so-called distributed time-delay systems. The above process yielding the transformed system (5.4) from the original one (5.1) is called *first-order model transformation*.

Subsequently, substitute (5.4) into $\dot{x}(t+\theta)$ in (5.2) as follows:

$$\begin{aligned} x(t-\tau) = x(t) - \int_{-\tau}^{0} \Bigg[&(A+B)x(t+\theta) \\ &- B\int_{-\tau}^{0} [\,Ax(t+\theta+\phi) + Bx(t-\tau+\theta+\phi)\,]d\phi\Bigg]d\theta. \end{aligned} \tag{5.5}$$

When (5.5) is embedded into the original system (5.1), we have

$$\begin{aligned} \dot{x}(t) = (A+B)x(t) - B\int_{-\tau}^{0} \Bigg[&(A+B)x(t+\theta) \\ &- B\int_{-\tau}^{0} [\,Ax(t+\theta+\phi) + Bx(t-\tau+\theta+\phi)\,]d\phi\Bigg]d\theta. \end{aligned} \tag{5.6}$$

The process which transforms the original system (5.1) to the distributed time-delay system (5.6) is this time called *second-order model transformation*.

The transformed systems (5.4) and (5.6) are utilized in order to derive some kind of stability conditions for the original one (5.1) by combining with the Lyapunov methods [24, 56, 57]. However, it should be emphasized that the transformed systems are not equivalent to the original system (5.1) in terms of stability. Namely, the transformed systems might provide conservative results. Let us explain this nature in the following.

Let $p_o(s)$ and $p_t(s)$ be the characteristic quasi-polynomials of the original system (5.1) and the transformed system (5.4) of the first type respectively. Then, letting $p_a(s)$ be the characteristic function of the additional dynamics, $p_t(s)$ is formulated as

$$p_t(s) = p_o(s)p_a(s), \tag{5.7}$$

where

$$p_a(s) = \det \left[I - \frac{1 - e^{-\tau s}}{s} B \right] \tag{5.8}$$

and $p_o(s) = \det[\, sI - A - Be^{-\tau s}\,]$. Thus, it can be realized that the additional dynamics arises only from B, i.e. the delay term. The roots of additional dynamics are called *additional eigenvalues*. Since the equation $p_a(s) = 0$ is equivalent to

$$1 - \lambda_i \frac{1 - e^{-\tau s}}{s} = 0, \quad i = 1, \cdots, n, \tag{5.9}$$

where λ_i, $i = 1, \cdots, n$ are the eigenvalues of B, they can be computed as the solutions of (5.9).

If the system (5.4) is stable, i.e. all of the roots of $p_t(s)$ lie in $\mathbf{C}^-$, it is necessary that the original system (5.1) and the additional dynamics are also stable because the roots of $p_o(s)$ and $p_a(s)$ must be all in $\mathbf{C}^-$. However, its converse is not true. Even though $p_t(s)$ is unstable, it is unclear whether $p_o(s)$ or $p_a(s)$ is unstable or both of them are unstable. It can be also seen that the stablity of $p_o(s)$ and $p_t(s)$ are equivalent only if under the assumption that $p_a(s)$ is stable. Conversely, the assumption that $p_a(s)$ is unstable immediately involves the result that $p_t(s)$ is unstable. Summarizing the above observations, the supposition that $p_a(s)$ is stable makes the stability of the transformed and original systems equivalent. On the other hand, when $p_a(s)$ is unstable, the transformed system (5.4) is also unstable but the stability of the original system (5.1) cannot be inferred from this fact, namely the model transformation makes no sense.

Let $p_{2t}(s)$ and $p_{2a}(s)$ be the characteristic functions of the transformed system (5.6) and the additional dynamics in terms of the second type respectively. In the similar way, $p_{2t}(s)$ is formed as

$$p_{2t}(s) = p_o(s) p_{2a}(s), \tag{5.10}$$

where $p_{2a}(s)$ is defined by

$$p_{2a}(s) = \det \left[I - \frac{1 - e^{-\tau s}}{s} B + \left(\frac{1 - e^{-\tau s}}{s} B \right)^2 \right]. \tag{5.11}$$

The additional eigenvalues of the second type are derived from the equations

$$\begin{aligned} 1 - \lambda_i e^{j\frac{\pi}{3}} \frac{1 - e^{-\tau s}}{s} &= 0, \quad i = 1, \cdots, n, \\ 1 - \lambda_i e^{-j\frac{\pi}{3}} \frac{1 - e^{-\tau s}}{s} &= 0, \quad i = 1, \cdots, n. \end{aligned} \tag{5.12}$$

It is obvious from (5.10) that a similar argument to the first type is maintained for the stability analysis of $p_{2t}(s)$, $p_o(s)$ and $p_{2a}(s)$.

Summing up this section, we arrive at the following statement which gives the motivation for the additional dynamics analysis [24–26, 47, 48, 50].

Theorem 5.1 The transformed system (5.4) induced by the first-order model transformation or (5.6) induced by the second-order model transformation is equivalent to the original time-delay system (5.1) in terms of stability if and only if the additional dynamics yielded from each transformation is stable.

We take note that the formulae (5.9) and (5.12) have suitable forms for applying the Lambert W function. In the subsequent section, the stability of the additional dynamics is investigated for each transformation by means of this function.

5.2 Stability Analysis by the Lambert W Function

5.2.1 Stability of the First Type Additional Dynamics

First, the additional dynamics induced by the first-order model transformation described by (5.9) is investigated. For the sake of simplicity, the subscript of λ_i is omitted in what follows, that is λ represents the arbitrary eigenvalues of B.

Suppose $-\lambda\tau \neq -1$. This means that any additional eigenvalues are not at the origin because (5.9) is vanished by $s = 0$ if and only if $-\tau\lambda = -1$. Thanks to this supposition, (5.9) can be modified as follows:

$$\begin{aligned} & 1 - \lambda\frac{1 - e^{-\tau s}}{s} = 0 && (5.13)\\ \Leftrightarrow\ & s - \lambda + \lambda e^{-\tau s} = 0 \\ \Leftrightarrow\ & (s - \lambda)e^{\tau s} = -\lambda \\ \Leftrightarrow\ & (s - \lambda)e^{\tau(s-\lambda)} = -\lambda e^{-\tau\lambda} \\ \Leftrightarrow\ & \tau(s - \lambda)e^{\tau(s-\lambda)} = -\tau\lambda e^{-\tau\lambda}. && (5.14) \end{aligned}$$

Using the definition of the Lambert W function (2.17) and (2.18), (5.14) can be further modified as

$$\begin{aligned} & \tau(s - \lambda) = W(-\tau\lambda e^{-\tau\lambda}) \\ \Leftrightarrow\ & s = \frac{1}{\tau}W(-\tau\lambda e^{-\tau\lambda}) + \lambda, && (5.15) \end{aligned}$$

which gives an explicit expression of the additional eigenvalues.

Remark 5.1 One should be cautious that even if (5.15) is satisfied for $s = 0$, this does not imply that (5.13) has the root $s = 0$ because of the assumption $s \neq 0$.

Now define a function $A_E : \mathbf{C} \to \mathbf{C}$ as

$$A_E(z) := W_0(ze^z) - z. \tag{5.16}$$

The following lemma elucidates a crucial feature of the function A_E.

Lemma 5.2 For any $z \in \mathbf{C}$, $\mathrm{Re}[\,A_E(z)\,] \geq 0$ holds true.

Proof Let $y = ze^z$. Note that this forms (2.17). Then, for a given $z \in \mathbf{C}$, there is a $k \in \mathbf{Z}$, where $\mathbf{Z}$ denotes a integer set, such that

$$z = W_k(y). \tag{5.17}$$

Replacing ze^z in (5.16) by y and z by $W_k(y)$, we have

$$A_E(y) = W_0(y) - W_k(y). \tag{5.18}$$

From Lemma 2.3,

$$\mathrm{Re}[\,W_0(y) - W_k(y)\,] \geq 0 \tag{5.19}$$

is fulfilled for any $y \in \mathbf{C}$ and $k \in \mathbf{Z}$. Connecting (5.18) with (5.19) yields the conclusion. $\square$

Lemma 5.2 yields the following stability condition for the additional dynamics of the first type.

Theorem 5.3 The additional dynamics induced by the first-order model transformation is stable if and only if the conditions (I) and (II) are both satisfied:

(I) $A_E(-\tau\lambda) = 0$,

(II) $-\tau\lambda \notin W_0(B_{C0})$.

Here, B_{C0} is as defined in (2.19).

Proof Reformulate (5.15) as

$$s_k = \frac{1}{\tau} W(-\tau\lambda e^{-\tau\lambda}) + \lambda, \quad k = 0, \pm 1, \cdots, \pm\infty \tag{5.20}$$

associated with the identity $A_E(-\tau\lambda) = \tau s_0$.
(Necessity): To prove by the contradiction, first assume $A_E(-\tau\lambda) \neq 0$. Then, $A_E(-\tau\lambda)$ lies in the complex close right half-plane excluding the origin from Lemma 5.2. Therefore, it turns out that the additional eigenvalue s_0 also lies in the right half-plane; thus the additional dynamics is unstable.

Now, under the assumption $A_E(-\tau\lambda) = 0$, i.e. $s_0 = 0$, further assume $-\tau\lambda \in W_0(B_{C0})$. Since $-\tau\lambda = W_0(-\tau\lambda e^{-\tau\lambda})$ due to $A_E(-\tau\lambda) = 0$, $-\tau\lambda e^{-\tau\lambda} \in B_{C0}$ is implied by the assumption $-\tau\lambda \in W_0(B_{C0})$. Furthermore, Remark 2.1 indicates

$$\begin{aligned} s_{-1} &= \frac{1}{\tau}W_{-1}(-\tau\lambda e^{-\tau\lambda}) + \lambda \\ &= \frac{1}{\tau}W_0^*(-\tau\lambda e^{-\tau\lambda}) + \lambda \\ &= s_0^* - \lambda^* + \lambda \\ &= -\lambda^* + \lambda \\ &= 2\mathrm{Im}[\,\lambda\,]j \notin \mathbf{C}^-, \end{aligned} \tag{5.21}$$

with (*) being the complex conjugate symbol. (5.21) involves instability of the additional dynamics and therefore the necessity is valid.

(Sufficiency): Assume that (I) and (II) are satisfied. Note that (I) is equivalent to $s_0 = 0$. Then, since $-\tau\lambda$ is contained in W_0, $-\tau\lambda e^{-\tau\lambda} \notin B_{C0}$ has to be required by (II). Recalling (2.22) of Lemma 2.3, $\mathrm{Re}[\,s_k\,] < \mathrm{Re}[\,s_0\,] = 0$, $k = \pm 1, \cdots, \pm\infty$ is true. Moreover, (II) ensures $-\tau\lambda \neq -1$ (see Figure 2.3), so that $s = 0$ is not a solution of (5.13) although $s_0 = 0$. Consequently, all of the additional eigenvalues have negative real parts and thus the theorem is proved. □

Remark 5.2 While $-\tau\lambda e^{-\tau\lambda} \notin B_{C0}$ guarantees (II) of Theorem 5.3, this is not a necessary condition for (II) due to the multi-valuedness of W. As a counter example, $-\tau\lambda \in W_1(B_{C0})$ obviously implies $-\tau\lambda e^{-\tau\lambda} \in B_{C0}$.

Noting that $A_E(-\tau\lambda) = 0$ can be rewritten as $-\tau\lambda = W_0(-\tau\lambda e^{-\tau\lambda})$, Corollary 5.4 below is a straightforward extension of Theorem 5.3.

Corollary 5.4 The additional dynamics induced by the first-order model transformation is stable if and only if

$$-\tau\lambda \in W_0(B_{C0}^c), \tag{5.22}$$

where B_{C0}^c denotes the complement of B_{C0}.

Remark 5.3 [47, Lemma 7] and [25, Theorem 3 and Corollary 4] can be derived from Corollary 5.4. In particular, it can be shown that there is no delay margin for stability except for it in [25][1].

[1]This fact has been proven in [25] apparently. However the proof was certainly dropped.

In the rest of this subsection, consider the robust stability of the first type additional dynamics. Let λ be uncertain parameter and prescribed by the polytope P_λ defined by

$$\lambda \in P_\lambda := \left\{ \sum_{i=1}^{n_\lambda} \gamma_i \lambda^i \;\middle|\; \sum_{i=1}^{n_\lambda} \gamma_i = 1,\, \gamma_1, \cdots, \gamma_{n_\lambda} \geq 0 \right\}, \tag{5.23}$$

where $\lambda^1, \cdots, \lambda^{n_\lambda} \in \mathbf{C}$. Because $W_0(B_{C0}^c)$ in Corollary 5.4 is a convex set as shown in the above part of Lemma 3.2, an extreme point result takes place.

Proposition 5.5 Suppose that the additional dynamics induced by the first-order model transformation has the uncertainty prescribed by $\lambda \in P_\lambda$. Then it is robustly stable if and only if Theorem 5.3 or Corollary 5.4 is fulfilled for $\lambda^1, \cdots, \lambda^{n_\lambda}$.

5.2.2 Stability of the Second Type Additional Dynamics

This subsection focuses on the stability of the second type additional dynamics. As is obvious from (5.12), the results are immediate by replacing λ in the previous subsection with $\lambda_i e^{j\frac{\pi}{3}}$ and $\lambda_i e^{-j\frac{\pi}{3}}$, $i = 1, \cdots, n$. Again let us drop the subscript of λ_i for simplicity.

Theorem 5.3 can be modified to the suitable form for the second type transformation.

Theorem 5.6 The additional dynamics induced by the second-order model transformation is stable if and only if all of the following conditions (I), (II), (III) and (IV) are satisfied:

(I) $A_E(-\tau\lambda e^{j\frac{\pi}{3}}) = 0$,

(II) $A_E(-\tau\lambda e^{-j\frac{\pi}{3}}) = 0$,

(III) $-\tau\lambda e^{j\frac{\pi}{3}} \notin W_0(B_{C0})$,

(IV) $-\tau\lambda e^{-j\frac{\pi}{3}} \notin W_0(B_{C0})$.

Remark 5.4 Likewise to Remark 5.2, $-\tau\lambda e^{\pm j\frac{\pi}{3}} e^{-\tau\lambda e^{\pm j\frac{\pi}{3}}} \notin B_{C0}$ is only sufficient but not necessary for $-\tau\lambda e^{\pm j\frac{\pi}{3}} \notin W_0(B_{C0})$.

The multiplier $e^{\pm j\frac{\pi}{3}}$ carries out the $\pm\pi/3$ rotation around the origin. The following result is evident from Corollary 5.4.

Corollary 5.7 The additional dynamics induced by the second-order model transformation is stable if and only if

$$-\tau\lambda \in W_0^{\frac{\pi}{3}}(B_{C0}^c) \cap W_0^{-\frac{\pi}{3}}(B_{C0}^c), \tag{5.24}$$

here $W_0^\theta(B_{C0}^c)$ stands for the θ rotation around the origin of $W_0(B_{C0}^c)$.

Remark 5.5 The second type additional dynamics has been studied in [26]. A similar assertion to Remark 5.3 can be made for the same reason.

Now switch to the robust stability investigation. Since $W_0^{\frac{\pi}{3}}(B_{C0}^c)$ and $W_0^{-\frac{\pi}{3}}(B_{C0}^c)$ are convex sets, $W_0^{\frac{\pi}{3}}(B_{C0}^c) \cap W_0^{-\frac{\pi}{3}}(B_{C0}^c)$ is also convex. If the same uncertainty is priscribed, a slight modification of Proposition 5.5 allows us to obtain the following criterion.

Proposition 5.8 Suppose that the additional dynamics induced by the second-order model transformation has the uncertainty prescribed by $\lambda \in P_\lambda$. Then it is robustly stable if and only if Theorem 5.6 or Corollary 5.7 is fulfilled for $\lambda^1, \cdots, \lambda^{n_\lambda}$.

5.3 Illustrative Examples

The first example demonstrates the robust stability condition obtained in the previous section.

Example 5.1 Let λ, one of the eigenvalues of B in the system (5.1), be uncertain parameter prescribed by the polytope

$$\lambda \in \left\{ \sum_{i=1}^{3} \gamma_i \lambda^i \;\middle|\; \sum_{i=1}^{3} \gamma_i = 1,\ \gamma_1, \gamma_2, \gamma_3 \geq 0 \right\},$$

where

$$\lambda^1 = 0.5 + j0.5,\ \lambda^2 = 0.5 - j0.5,\ \lambda^3 = -4.$$

The robust stability of the above uncertain additional dynamics in terms of the first-order transformation is checked with stability delay margin using Proposition 5.5 in the following.

According to Proposition 5.5, depict graphs of $|A_E(-\tau\lambda^1)|$, $|A_E(-\tau\lambda^2)|$ and $|A_E(-\tau\lambda^3)|$ versus time-delay τ as in Figures 5.1–5.3 using the "ProductLog" function of Mathematica. It is verified that the additional dynamics is robustly stable for $0 < \tau < 1.57$ from these

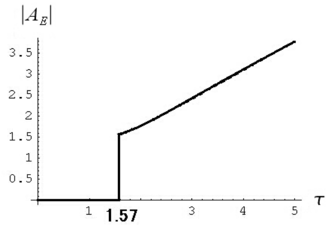

Figure 5.1: $|A_E(-\tau\lambda^1)|$ against τ.

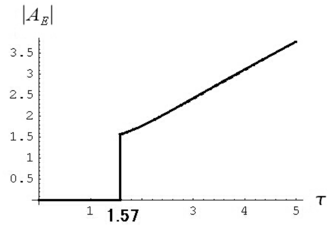

Figure 5.2: $|A_E(-\tau\lambda^2)|$ against τ.

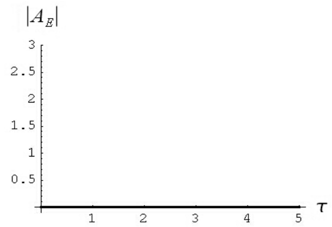

Figure 5.3: $|A_E(-\tau\lambda^3)|$ against τ.

figures. When one intends to find out stability delay margin like this example, one can be uncareful about the condition (II) of Theorem 5.3 since it is inevitably broken at the stability limit of τ.

The second example illustrates an inclusion relation between the first-order and second-order transformations.

Example 5.2 Let $\tau = 1$ and $\lambda = -3$. For this case,

$$\begin{aligned} &|A_E(-\tau\lambda)| = 0, \\ &-\tau\lambda e^{-\tau\lambda} = 60.26 \notin B_{C0}, \\ &|A_E(-\tau\lambda e^{j\frac{\pi}{3}})| = 4.42 \end{aligned}$$

are observed. According to Theorem 5.3 and 5.7 and reminding Remark 5.2, this additional dynamics is stable in the sense of the first type but unstable in the second type.

For the same τ, change λ value to $\lambda = 1.1$. Then, we have

$$\begin{aligned} &|A_E(-\tau\lambda)| = 0.19, \\ &|A_E(-\tau\lambda e^{j\frac{\pi}{3}})| = 0, \\ &-\tau\lambda e^{j\frac{\pi}{3}} e^{-\tau\lambda e^{j\frac{\pi}{3}}} = -0.63 - j0.06 \notin B_{C0}, \\ &|A_E(-\tau\lambda e^{-j\frac{\pi}{3}})| = 0, \\ &-\tau\lambda e^{-j\frac{\pi}{3}} e^{-\tau\lambda e^{-j\frac{\pi}{3}}} = -0.63 + j0.06 \notin B_{C0}. \end{aligned}$$

Again according to Theorem 5.3 and 5.7 and Remark 5.4 this time, the additional dynamics of the first type is unstable but the second type stable.

Furthermore, if $\lambda = -1$ for the same τ,

$$\begin{aligned} &|A_E(-\tau\lambda)| = 0, \\ &-\tau\lambda e^{-\tau\lambda} = 2.72 \notin B_{C0}, \\ &|A_E(-\tau\lambda e^{j\frac{\pi}{3}})| = 0, \\ &-\tau\lambda e^{j\frac{\pi}{3}} e^{-\tau\lambda e^{j\frac{\pi}{3}}} = -0.55 + j1.55 \notin B_{C0}, \\ &|A_E(-\tau\lambda e^{-j\frac{\pi}{3}})| = 0, \\ &-\tau\lambda e^{-j\frac{\pi}{3}} e^{-\tau\lambda e^{-j\frac{\pi}{3}}} = -0.55 - j1.55 \notin B_{C0} \end{aligned}$$

are satisfied, namely the first type and second type transformations are both stable.

The above observations illustrate that the stability regions with respect to λ for the two types of transformation have an intersection, but they do not strictly include each other. This fact can be visualized as in Figure 5.4. This figure also indicates that the stability region for the second transformation is however almost included by the first type.

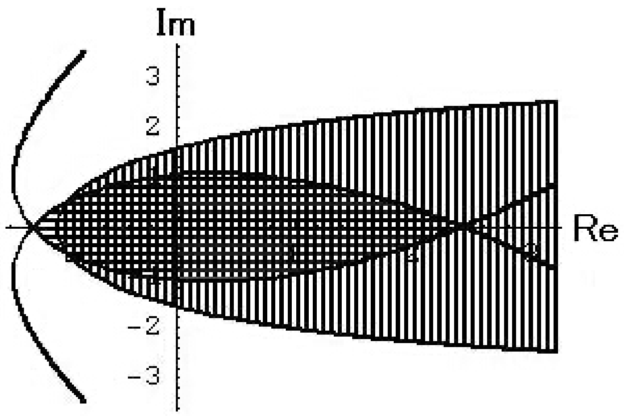

Figure 5.4: Area with vertical stripes: $W_0(B^c_{C0})$ and horizontal stripes: $W_0^{\frac{\pi}{3}}(B^c_{C0}) \cap W_0^{-\frac{\pi}{3}}(B^c_{C0})$.

5.4 Concluding Remarks

In this chapter, additional dynamics, which is induced by model transformations of linear time-delay systems, has been investigated by means of the Lambert W function. This function could be well suited to the stability analysis of the additional dynamics, because their characteristic functions could be algebraically solved by this function. As a result, stability and robust stability conditions were given in terms of the first-order and second-order model transformations.

As an alternative model transformation, there is a so-called neutral model transformation [26, 48]. Since this additional dynamics is analogous to the first-order, therefore the second-order transformation, the Lambert W function would give the similar results to those in Section 5.2 immediately. In general, these three transformations can be generalized into the framework of a parametrized model transformation [26, 48]. This type of transformation can be analyzed by the Lambert W function as well and consequently general results could be provided in a trivial way.

The model transformations are introduced for utilizing some types of the Lyapunov functions or functionals. However, the model transformations have not been used so much in the recent tendency since derivation techniques of stability conditions were greatly improved in [80]. Nowadays, less conservative results are being published one after another. Nevertheless, once one applies stability criteria based on the model transformations, one can enjoy handy stability test schemes, and then the additional dynamics analysis would supply useful information on their conservativeness or usefulness.

Chapter 6

Conclusion

The main contribution of this thesis is to develop robust stability criteria and stabilization techniques of linear time-delay systems via the Lambert W function. In the robust stability analysis of a class of linear single delay systems explored in Chapter 3, the two types of simplifications of robust stability conditions have been derived: one is the extreme point result, the other is the boundary implication. Either of them can specify the crucial positions in the prescribed untertain parameter regions for the robust stability. More specifically, the extreme point result tells us that the robust stability for the box-type and the sector-type uncertainties in the coefficients is determined by only a few boundary extreme points of them. Given more general uncertainties, the boundary implication of stability takes place, which addresses that the critical points for the robust stability exist on the boundary of the uncertain regions. While they give us the effective robust stability check schemes, the available system class is limited by the applicability of the scalar Lambert W function, corresponding to simultaneously triangularizable systems. To relax this limitation is a currently open issue.

In Chapter 4, a new pole placement technique using the Lambert W function has been developed. And then combining it with the decoupling control of [87], a controller design procedure was also devised as summarized in Algorithm 4.1. The proposed control scheme firstly constructs a fundamental characteristic quasi-polynomial based on the most simple ones by the decoupling controllers, and then allocates the desired poles following the guideline Theorem 4.5. In contrast to the finite spectrum assignment introduced in Section 2.4, an infinite number of poles are stayed in the closed-loop systems. Meanwhile, by queueing the uncontrolled poles on the left or under of the arbitrary assigned poles, stabilization can be carried out by a finite number of poles in spite of the existence of infinitely many poles. For avoiding predictive control, non-predictive decoupling controllers were adopted. Thanks to these types of controllers, integral terms are not required in the feedback loops and therefore one does not encounter the implimentation problems for such as the finite spectrum assignment based on spectrally controllable systems. Furthermore, due to the delay terms left in the closed-loop systems, one can easily compute stability delay margin of them using the Lambert W function.

Finally in Chapter 5, additional dynamics induced by model transformations of linear time-delay systems has been investigated via the Lambert W function. This function can solve the characteristic equations of the additional dynamics and give explicit expressions of the additional eigenvalues. In particular, the first-order and second-order additional dynamics have been investigated in this chapter, and as a result handy stability and robust stability criteria for them were obtained due to the contribution of the W function. It should be stressed that the obtained results can be immediately modified to the suitable forms for a neutral model transformation and general parametrized transformation.

The advantage of using the Lambert W function is to be able to give explicit expressions of the characteristic roots of linear time-delay systems. The characteristic roots represented by this function can be easily computed using the computer software Mathematica, Maple or Matlab. Furthermore, by virtue of Lemma 2.3, one can always grasp the crucial root for stablity and this fact makes the stability features considerably transparent. It is not so long since this function was reborn in [15], and therefore it would still have some room for improvement. The author expects that stability analysis and stabilization techniques using this function could be further developed by prospective researchers and the Lambert W function approach would be qualified as one of fundamental approaches of time-delay system analysis.

Appendix A

Proof of Lemma 2.3

Lemma A.1 below, with which the proof of Lemma 2.3 begins, asserts nonsingularity of each branch of the Lambert W function in the complement of the branch cuts.

Lemma A.1 $W_0(z)$ is analytic in the complement of B_{C0}, say B_{C0}^c where $(^c)$ signifies a complement of a set. $W_k(z)$, $k = \pm 1, \cdots, \pm\infty$ are also analytic in B_C^c.

Proof Let $V(w) := we^w$, $w \in \mathbf{C}$. $V(w)$ is an analytic function and $dV(w)/dw = (1+w)e^w$ holds. Since $dV(w)/dw \neq 0$ for $w \neq -1$, there is an analytic inverse function of $V(w)$ in a suitable neighborhood of $w \neq -1$ [3]. Then, $W_0(z)$ is the analytic inverse function of $V(w)$ in $W_0(B_{C0}^c)$. $W_k(z)$, $k = \pm 1, \cdots, \pm\infty$ are also the analytic inverse functions in $W_k(B_C^c)$, $k = \pm 1, \cdots, \pm\infty$ respectively. □

Hereafter, consider the mapping of the circle C_r in (2.24). The subsequent two lemmas shed light on monotonicities of the Lambert W function on the circle C_r concerning the real part.

Lemma A.2 $\mathrm{Re}[\, W_0(re^{j\theta})\,]$ is a monotone increasing function of $\theta \in (-\pi, 0]$ and a monotone decreasing function of $\theta \in [0, \pi]$. $\mathrm{Re}[\, W_k(re^{j\theta})\,]$, $k = -1, \cdots, -\infty$ are monotone increasing functions of $\theta \in (-\pi, \pi]$. $\mathrm{Re}[\, W_k(re^{j\theta})\,]$, $k = 1, \cdots, \infty$ are monotone decreasing functions of $\theta \in (-\pi, \pi]$.

Proof Let $\theta \neq \pi$. Then, since $re^{j\theta} \notin B_{C0}$, $W_0(re^{j\theta})$ is analytic from Lamma A.1. Thus

$$\mathrm{Re}\left[\frac{dW_0(re^{j\theta})}{d\theta}\right] = \mathrm{Re}\left[\frac{W_0(re^{j\theta})}{1 + W_0(re^{j\theta})}j\right] \tag{A.1}$$

is obtained. Setting $W_0(re^{j\theta}) = \xi_0 + j\eta_0$, (A.1) is written as

$$\frac{-\eta_0}{(1+\xi_0)^2 + \eta_0^2}. \tag{A.2}$$

Since $\eta_0 < 0$ for $\theta \in (-\pi, 0)$ and $\eta_0 > 0$ for $\theta \in (0, \pi)$ (see Figure 2.3), (A.2) > 0 and (A.2) < 0 are fulfilled respectively. This shows that $\mathrm{Re}[\,W_0(re^{j\theta})\,]$ is monotone increasing in $\theta \in (-\pi, 0)$ and monotone decreasing in $\theta \in (0, \pi)$. Moreover, monotonicity is preserved at $\theta = 0$ and π becasue of continuity of $\mathrm{Re}[\,W_0(re^{j\theta})\,]$ at these points.

In the same way, $W_k(re^{j\theta})$, $k = \pm 1, \cdots, \pm\infty$ are analytic for $re^{j\theta} \notin B_C$ by Lemma A.1 and we have

$$\mathrm{Re}\left[\frac{dW_k(re^{j\theta})}{d\theta}\right] = \frac{-\eta_k}{(1+\xi_k)^2 + \eta_k^2}, \quad k = \pm 1, \cdots, \pm\infty, \tag{A.3}$$

with $W_k(re^{j\theta}) = \xi_k + j\eta_k$, $k = \pm 1, \cdots, \pm\infty$. Since $\eta_k < 0$ for W_k, $k = -1, \cdots, -\infty$ (see Figure 2.4), (A.3) > 0 is satisfied, that is $\mathrm{Re}[\,W_k(re^{j\theta})\,]$, $k = -1, \cdots, -\infty$ are monotone increasing in $\theta \in (-\pi, \pi)$. Similarly, since $\eta_k > 0$ for W_k, $k = 1, \cdots, \infty$ (see Figure 2.5), (A.3) < 0 is derived and thus $\mathrm{Re}[\,W_k(re^{j\theta})\,]$, $k = 1, \cdots, \infty$ are monotone decreasing in $\theta \in (-\pi, \pi)$. Thereby, continuity of $\mathrm{Re}[\,W_k(re^{j\theta})\,]$, $k = \pm 1, \cdots, \pm\infty$ at $\theta = \pi$ keeps the monotonicity and concludes the proof. □

Lemma A.3 $\mathrm{Re}[\,W_k(re^{j\theta})\,]$, $k = \pm 1, \cdots, \pm\infty$ are monotone increasing functions of r.

Proof Let $\theta \neq \pi$. Then, $re^{j\theta} \notin B_C$ and Lemma A.1 endorses that $W_k(re^{j\theta})$, $k = \pm 1, \cdots, \pm\infty$ are analytic. Therefore, for $k = \pm 1, \cdots, \pm\infty$

$$\mathrm{Re}\left[\frac{dW_k(re^{j\theta})}{dr}\right] = \mathrm{Re}\left[\frac{W_k(re^{j\theta})}{r(1 + W_k(re^{j\theta}))}\right] \tag{A.4}$$

are hold true. Setting $W_k(re^{j\theta}) = \xi_k + j\eta_k$, $k = \pm 1, \cdots, \pm\infty$, (A.4) is written as

$$\frac{\xi_k^2 + \xi_k + \eta_k^2}{r((1+\xi_k)^2 + \eta_k^2)}, \quad k = \pm 1, \cdots, \pm\infty. \tag{A.5}$$

If $\xi_k = \mathrm{Re}[\,W_k(re^{j\theta})\,] < -1$, (A.5) > 0 holds. Namely, in this case, $\mathrm{Re}[\,W_k(re^{j\theta})\,]$, $k = \pm 1, \cdots, \pm\infty$ are monotone increasing with respect to r. Moreover, thanks to continuity of $\mathrm{Re}[\,W_k(re^{j\theta})\,]$, $k = \pm 1, \cdots, \pm\infty$ at $\theta = \pi$, monotonicity is preserved at this point.

Then consider the case where $\xi_k = \mathrm{Re}[\,W_k(re^{j\theta})\,] \geq -1$, $k = \pm 1, \cdots, \pm\infty$. To grasp the sign of (A.5), let us observe the sign of the numerator of (A.5) on the boundary of W_0, that is, $\xi = -\eta/\tan\eta$, $\eta \in (-\pi, \pi)$ [15]. We have

$$\begin{aligned} \xi^2 + \xi + \eta^2 &= \frac{\eta^2}{\tan^2\eta} - \frac{\eta}{\tan\eta} + \eta^2 \\ &= \frac{\eta}{\tan^2\eta\cos^2\eta}(\eta - \sin\eta\cos\eta). \end{aligned} \tag{A.6}$$

It is easy to see (A.6) ≥ 0. Now fix $\xi = \xi_c \geq -1$. Then, η takes non-negative and non-positive values doubly; let these values be η_c^+ and η_c^- where $\eta_c^+ = -\eta_c^-$ respectively. Note that $\eta_c^+ = \eta_c^- = 0$ when $\xi_c = -1$. Letting $\xi_k = \xi_c$, $k = \pm 1, \cdots, \pm\infty$, we have $\eta_k > \eta_c^+$ for $k = +1, \cdots, +\infty$ and $\eta_k < \eta_c^-$ for $k = -1, \cdots, -\infty$ (see Figures 2.2, 2.4 and 2.5). Therefore, (A.5) > 0 is satisfied, namely $\mathrm{Re}[\, W_k(re^{j\theta})\,]$, $k = \pm 1, \cdots, \pm\infty$ are monotone increasing with respect to r. Since $\mathrm{Re}[\, W_k(re^{j\theta})\,]$, $k = \pm 1, \cdots, \pm\infty$ are continuous at $\theta = \pi$, monotonicity is carried over to this point and the proof is completed. □

Now we are at the stage of proving Lemma 2.3 based on Lemmas A.2 and A.3.

Proof (Lemma 2.3) Since $\mathrm{Re}[\, W_0(0)\,] = 0$ and $\lim_{z\to 0} \mathrm{Re}[\, W_k(z)\,] = -\infty$, $k = \pm 1, \cdots, \pm\infty$, the lemma is immediate for $z = 0$, and thus assume $z \neq 0$.

Let N be an integer less than or equal to -2. Then C_r where $r > 0$ crosses the branch cuts of both W_N and W_{N+1}, so that $W_N(C_r)$ is connected to $W_{N+1}(C_r)$ continuously and

$$\mathrm{Re}[\, W_N(re^{j\pi})\,] = \lim_{\theta\to -\pi} \mathrm{Re}[\, W_{N+1}(re^{j\theta})\,] \tag{A.7}$$

holds. (A.7) combining with Lemma A.2 implies $\mathrm{Re}[\, W_N(re^{j\theta})\,] < \mathrm{Re}[\, W_{N+1}(re^{j\theta})\,]$ for any $N \leq -2$ that yields

$$\max_{k=-1,\cdots,-\infty} \mathrm{Re}[\, W_k(re^{j\theta})\,] = \mathrm{Re}[\, W_{-1}(re^{j\theta})\,] \tag{A.8}$$

for any $r > 0$ and $\theta \in (-\pi, \pi]$. By similar reasoning,

$$\max_{k=1,\cdots,\infty} \mathrm{Re}[\, W_k(re^{j\theta})\,] = \mathrm{Re}[\, W_1(re^{j\theta})\,] \tag{A.9}$$

is fulfilled for any $r > 0$ and $\theta \in (-\pi, \pi]$.

Now assume $r \geq 1/e$. Then, since C_r crosses the branch cut B_{C0}, $W_{-1}(C_r)$ is connected to $W_0(C_r)$ continuously and

$$\mathrm{Re}[\, W_{-1}(re^{j\pi})\,] = \lim_{\theta\to -\pi} \mathrm{Re}[\, W_0(re^{j\theta})\,] \tag{A.10}$$

is met. Lemma A.2 further reveals that

$$\inf_{\theta\in(-\pi,\pi]} \mathrm{Re}[\, W_0(re^{j\theta})\,] = \mathrm{Re}[\, W_0(re^{j\pi})\,], \tag{A.11}$$

$$\sup_{\theta\in(-\pi,\pi]} \mathrm{Re}[\, W_{-1}(re^{j\theta})\,] = \mathrm{Re}[\, W_{-1}(re^{j\pi})\,]. \tag{A.12}$$

Meanwhile, symmetry of the image of W_0 with respect to the real axis (Remark 2.1) entails

$$\mathrm{Re}[\, W_0(re^{j\pi})\,] = \lim_{\theta\to -\pi} \mathrm{Re}[\, W_0(re^{j\theta})\,]. \tag{A.13}$$

From (A.13) and (A.11),

$$\inf_{\theta\in(-\pi,\pi]} \mathrm{Re}[\, W_0(re^{j\theta})\,] = \lim_{\theta\to -\pi} \mathrm{Re}[\, W_0(re^{j\theta})\,] \tag{A.14}$$

is true and adjusting (A.12) and (A.14) to (A.10) results in

$$\sup_{\theta \in (-\pi, \pi]} \mathrm{Re}[\, W_{-1}(re^{j\theta})\,] = \inf_{\theta \in (-\pi, \pi]} \mathrm{Re}[\, W_0(re^{j\theta})\,], \tag{A.15}$$

that is

$$\mathrm{Re}[\, W_{-1}(re^{j\theta})\,] = \mathrm{Re}[\, W_0(re^{j\theta})\,] \tag{A.16}$$

for any $r \geq 1/e$ and $\theta = \pi$, i.e. $re^{j\theta} \in B_{C0}$ corresponding to the case (2.23) since (A.11), (A.12) and (A.15) are satisfied and

$$\mathrm{Re}[\, W_{-1}(re^{j\theta})\,] < \mathrm{Re}[\, W_0(re^{j\theta})\,] \tag{A.17}$$

for any $r \geq 1/e$ and $\theta \in (-\pi, \pi)$ corresponding to the case (2.22). On the other hand, $W_1(C_r)$ is connected to $W_0(C_r)$ and

$$\lim_{\theta \to -\pi} \mathrm{Re}[\, W_1(re^{j\theta})\,] = \mathrm{Re}[\, W_0(re^{j\pi})\,] \tag{A.18}$$

is obtained. This time, Lemma A.2 provides

$$\sup_{\theta \in (-\pi, \pi]} \mathrm{Re}[\, W_1(re^{j\theta})\,] = \lim_{\theta \to -\pi} \mathrm{Re}[\, W_1(re^{j\theta})\,], \tag{A.19}$$

and therefore

$$\mathrm{Re}[\, W_1(re^{j\theta})\,] < \mathrm{Re}[\, W_0(re^{j\theta})\,] \tag{A.20}$$

for any $r \geq 1/e$ and $\theta \in (-\pi, \pi]$ because assembling (A.19), (A.18) and (A.11) involves

$$\sup_{\theta \in (-\pi, \pi]} \mathrm{Re}[\, W_1(re^{j\theta})\,] = \lim_{\theta \to -\pi} \mathrm{Re}[\, W_1(re^{j\theta})\,] = \inf_{\theta \in (-\pi, \pi]} \mathrm{Re}[\, W_0(re^{j\theta})\,]. \tag{A.21}$$

We now turn to the case of $r < 1/e$. In this case, C_r does not cross the branch cut B_{C0}. As a result, $W_0(C_r)$ is separated from the other branches (see also Figure 2.6) and we cannot follow the above discussion. Due to Lemma A.2 and the fact that $\mathrm{Re}[\, W_{-1}(1/e \cdot e^{j\pi})\,] = -1$ and $\lim_{\theta \to -\pi} \mathrm{Re}[\, W_1(1/e \cdot e^{j\theta})\,] = -1$, we have $\mathrm{Re}[\, W_{\pm 1}(1/e \cdot e^{j\theta})\,] \leq -1$ for $\theta \in (-\pi, \pi]$ where $W_{\pm 1}$ represents W_1 and W_{-1}. Lemma A.3 deduces a further limitation $\mathrm{Re}[\, W_{\pm 1}(re^{j\theta})\,] < -1$ for all $r < 1/e$ and $\theta \in (-\pi, \pi]$. Whereas, notice that $\mathrm{Re}[\, W_0(re^{j\theta})\,] > -1$ for $\theta \in (-\pi, \pi]$ in the case of $r < 1/e$ (see Figure 2.3). Consequently,

$$\mathrm{Re}[\, W_{\pm 1}(re^{j\theta})\,] < \mathrm{Re}[\, W_0(re^{j\theta})\,] \tag{A.22}$$

follows for arbitrary $r < 1/e$ and $\theta \in (-\pi, \pi]$.

Lemma 2.3 can now be concluded by gathering (A.8), (A.9), (A.17), (A.20) and (A.22) for (2.22) and (A.8), (A.9), (A.16) and (A.20) for (2.23). □

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