# Variational Formulas under $C^2$-movements

Dedicated to Professor Tatuo Fuji'i'e on his sixtieth birthday

By

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**Abstract**

We shall investigate the change of function-theoretic quantities as a Riemann surface deforms smoothly. As a smooth deformation, we consider Riemann surfaces with conformal structures decided by Beltrami differentials depending differentiably on parameters. Function-theoretic quantities are often represented by meromorphic differentials with specific kind of boundary behavior. We show here that their differentials vary differentiably and obtain variational formulas of function-theoretic quantities. We take up a recovering deformation as a quasiconformal deformation and apply our results.

## 1. Introduction

We have been investigated quasiconformal deformations of Riemann surfaces whose conformal structures are decided by Beltrami differentials holomorphically depending on a parameter and given variational formulas of various function-theoretic quantities, which are represented as inner products of meromorphic differentials with certain boundary behaviors. In this paper we consider them in the same frame but under weaker conditions. As the quasiconformal deformation we take up Riemann surfaces with conformal structures given by Beltrami differentials which have the second Frèchet derivatives with respect to a parameter. We simply call this a $C^2$-movement. We first show that meromorphic differentials with certain boundary behavior have the second Frèchet derivatives under a $C^2$-movement. It leads us to second variational formulas of various quantities. They have the same form as Taniguchi gave for Green functions. At last we deal with recovering deformations as Schiffer's interior variations or pinching deformations.

## 2. $C^2$-movements of Riemann Surfaces

Let $R$ be an arbitrary Riemann surface and consider Beltrami differentials

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$\mu(z, t)\frac{d\bar{z}}{dz}$ on $R$ with a real parameter $t=(t_1, \ldots, t_n)$ varying in a domain about $0\in R^n$. From the Beltrami coefficient $\mu(z, t)$ we get another Riemann surface $R^t$ with the Riemannian metric $ds=\lambda(z)|dz+\mu(z, t)d\bar{z}|$. We obtain a family of Riemann surfaces $\{R^t\}$ and call it a $C^2$-movement if the Beltrami differentials satisfy the following.

i. $\mu(z,t)$ is measurable, $\mu(z,0)=0$ and $\|\mu(z,t)\|=\text{ess sup } |\mu(z,t)|<1$,

ii. There exist Beltrami coefficients $\mu_i(z,t)$, $\mu_{ij}(z,t)$ such that

$$\lim_{h_i\to 0}\frac{1}{h_i}\|\mu(z, t+h_i)-\mu(z,t)-h_i\mu_i(z,t)\|=0,$$

$$\lim_{h_j\to 0}\frac{1}{h_j}\|\mu_i(z, t+h_j)-\mu_i(z,t)-h_j\mu_{ij}(z,t)\|=0,$$

where $\mu_i(z,t)$, $\mu_{ij}(z,t)$ are bounded measurable and for real $h_i$ and $(0, \ldots, h_i, \ldots, 0)\in \boldsymbol{R}^n$ the same notation $h_i$ is used for convenience. Let $f_t$ be the quasiconformal homeomorphism from $R$ to $R^t$ with the Beltrami coefficient $\mu(z,t)$. We express $f_t$ sometimes as $\zeta=f_t(z)$ in terms of respective generic local parameters $z$ and $\zeta$ of $R$ and $R^t$. Then $\mu(z,t)=\zeta_{\bar{z}}/\zeta_z$. The composition mapping $f_{t+\tau}\circ f_t^{-1}$ becomes a quasiconformal mapping from $R^t$ to $R^{t+\tau}$ whose Beltrami coefficient is

$$\nu(\zeta,\tau)=\frac{\mu(z,t+\tau)-\mu(z,t)}{1-\overline{\mu(z,t)}\mu(z,t+\tau)}\frac{\zeta_z}{\overline{\zeta_z}}.$$

### 3. Frèchet Derivatives of Meromorphic Differentials with Boundary Behavior

Let $\Lambda$ be the real Hilbert space of square integrable complex differentials whose inner product is given by

$$<\omega,\sigma> = \text{Real part of } \iint\omega\wedge{}^*\bar{\sigma}=\text{Re } (\omega,\sigma),$$

where ${}^*\sigma$ denotes the harmonic conjugate differential of $\sigma$ and $\bar{\sigma}$ denotes the complex conjugate of $\sigma$. Let $\Lambda_{eo}$ be the completion of the class which consists of the differentials of complex valued $C^\infty$-functions with compact supports and $\Gamma_h$ the subspace of real harmonic differentials. For a subspace $\Gamma_x$ of $\Gamma_h$ we denote the orthogonal complement in $\Gamma_h$ by $\Gamma_x^\perp$, by ${}^*\Gamma_x$ the space of harmonic conjugate differentials. For a differential $\omega$ in $\Lambda(R)$, $\omega\circ f_t^{-1}$ denotes the pull back by $f_t^{-1}$ of $\omega$. Let $\Gamma_x(R^t)$ be the orthogonal projection to $\Gamma_h(R^t)$ from $\{\omega\circ f_t^{-1};\ \omega\in\Gamma_x(R)\}$. Set $\Lambda_x(R)=\Gamma_x(R)+i{}^*\Gamma_x(R)^\perp$ and $\Lambda_x(R^t)=\Gamma_x(R^t)+i{}^*\Gamma_x(R^t)^\perp$.

Let the support of a Beltrami coefficient $\mu(z,t)$ have no intersection with a neighborhood $V$ of the poles of a meromorphic differential $\phi^0$ and assume that there exists a meromorphic differential $\phi^t$ on $R^t$ such that $\phi^t\circ f_t-\phi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$. Now we recall the following Lemma.[3)]

Lemma 1. For differentials $\omega$, $\sigma$ on $R$

$$(\omega\circ f_t^{-1}, -{}^*(({}^*\sigma)\circ f_t^{-1}))_{R^t}=(\omega,\sigma)_R.$$

By this lemma, if $\sigma\in{}^*\Gamma_x(R)^\perp$, for any $\omega\in\Gamma_x(R^t)$

$$(\omega, {}^*(\sigma\circ f_t^{-1}))_{R^t}=(\omega\circ f_t, {}^*\sigma)_R=0.$$

The $\sigma\circ f_t^{-1}$ is closed and $\sigma\circ f_t^{-1}\in{}^*\Gamma_x(R^t)^\perp+\Gamma_{eo}(R^t)$. Therefore if $\omega\in\Lambda_x(R)+\Lambda_{eo}(R)$, then $\omega\circ f_t^{-1}\in\Lambda_x(R^t)+\Lambda_{eo}(R^t)$. If the other $\psi^t$ satisfies $\psi^t\circ f_t-\phi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$, then

$$\phi^t\circ f_t-\psi^t\circ f_t\in\Lambda_x(R)+\Lambda_{eo}(R) \text{ and } \phi^t-\psi^t\in\Lambda_x(R^t).$$

It follows that

$$\|\phi^t-\psi^t\|^2=<\phi^t-\psi^t, \mathrm{i}^*(\phi^t-\psi^t)>=0.$$

Hence the $\phi^t$ is uniquely determined on $R^t$. This $\phi^t$ is differentiable with respect to $t$ in the following sense.

Theorem 1. There exists a differential $\phi_i^t\in\Lambda_x(R^t)+\Lambda_{eo}(R^t)$ such that

$$\lim_{h_i\to 0}\frac{1}{h_i}\|\phi^{t+h_i}\circ f_{t+h_i}\circ f_t^{-1}-\phi^t-h_i\phi_i^t\|=0.$$

Proof. Set

$$\omega(h_i)=\{\phi^{t+h_i}\circ f_{t+h_i}\circ f_t^{-1}+\mathrm{i}^*(\phi^{t+h_i}\circ f_{t+h_i}\circ f_t^{-1})\}/2$$
$$\sigma(h_i)=\{\phi^{t+h_i}\circ f_{t+h_i}\circ f_t^{-1}-\mathrm{i}^*(\phi^{t+h_i}\circ f_{t+h_i}\circ f_t^{-1})\}/2$$

Since $\omega(h_i)+\sigma(h_i)-\phi^t\in\Lambda_x(R^t)+\Lambda_{eo}(R^t)$, we have

$$\begin{aligned}0&=<\omega(h_i)+\sigma(h_i)-\phi^t, \mathrm{i}^*(\omega(h_i)+\sigma(h_i)-\phi^t)>\\&=<\omega(h_i)-\phi^t+\sigma(h_i), \omega(h_i)-\phi^t-\sigma(h_i)>,\end{aligned}$$

and $\|\omega(h_i)-\phi^t\|=\|\sigma(h_i)\|$.

From $\sigma(h_i)=\omega(h_i)\nu(\zeta, h_i)\dfrac{d\bar\zeta}{d\zeta}$, we obtain

$$\|\omega(h_i)\|_{R^t-V^t}\leqq\frac{1}{1-\|\nu(\zeta, h_i)\|}\|\phi^t\|_{R^t-V^t} \qquad (V^t=f_t^{-1}(V)),$$

and

$$\|\omega(h_i)-\phi^t\|=\|\sigma(h_i)\|\leqq\frac{\|\nu(\zeta, h_i)\|}{1-\|\nu(\zeta, h_i)\|}\|\phi^t\|_{R^t-V^t}$$

Write $\varepsilon_i(h_i)=\mu(z, t+h_i)-\mu(z, t)-h_i\mu_i(z, t)$ and $\nu_i(\zeta)=\dfrac{\mu_i(z, t)}{1-|\mu(z, t)|^2}\dfrac{\zeta_z}{\bar\zeta_z}$.

Then we have

$$\begin{aligned}&\lim_{h_i\to 0}\left\|\frac{\nu(\zeta, h_i)}{h_i}-\nu_i(\zeta)\right\|\\&=\lim_{h_i\to 0}\left\|\frac{(1-|\mu(z, t)|^2+h_i\overline{\mu(z, t)}\mu_i(z, t))\varepsilon_i(h_i)/h_i+h_i\overline{\mu(z, t)}\mu_i(z, t)^2}{(1-|\mu(z, t)|^2-\overline{\mu(z, t)}(h_i\mu_i(z, t)+\varepsilon_i(h_i)))(1-|\mu(z, t)|^2)}\frac{\zeta_z}{\bar\zeta_z}\right\|\\&=0.\end{aligned}$$

It follows that

$$\begin{aligned}&\lim_{h_i\to 0}\left\|\frac{\sigma(h_i)}{h_i}-\phi^t\nu_i(\zeta)\frac{d\bar\zeta}{d\zeta}\right\|\\&\leqq\lim_{h_i\to 0}\left\{\left\|(\omega(h_i)-\phi^t)\frac{\nu(\zeta, h_i)}{h_i}\frac{d\bar\zeta}{d\zeta}\right\|+\|\phi^t\|_{R^t-V^t}\left\|\frac{\nu(\zeta, h_i)}{h_i}-\nu_i(\zeta)\right\|\right\}\\&=\lim_{h_i\to 0}\frac{\|\nu(\zeta, h_i)\|}{1-\|\nu(\zeta, h_i)\|}\|\phi^t\|_{R^t-V^t}\left\|\frac{\nu(\zeta, h_i)}{h_i}\right\|=0.\end{aligned}$$

On the other hand,

$$\left\|\frac{\omega(h_i)-\phi^t}{h_i}-\frac{\omega(k_i)-\phi^t}{k_i}\right\|=\left\|\frac{\sigma(h_i)}{h_i}-\frac{\sigma(k_i)}{k_i}\right\|.$$

Hence $\{\sigma(h_i)/h_i\}$ and $\{(\omega(h_i)-\phi^t)/h_i\}$ are Cauchy sequences. Since $(\omega(h_i)+\sigma(h_i)-\phi^t)/h_i$ belongs to $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$, there exists a

$$\phi_i{}^t=\lim_{h_i\to 0}\frac{1}{h_i}(\omega(h_i)+\sigma(h_i)-\phi^t)\in\Lambda_x(R^t)+\Lambda_{eo}(R^t).$$

This is the differential in the assertion.

Remark 1. We have the following representation:

$$\phi_i{}^t-\mathrm{i}^*\phi_i{}^t=2\underline{\phi}_i{}^{t,1}d\bar{\zeta}=2\lim_{h_i\to 0}\frac{\sigma(h_i)}{h_i}$$

$$=2\phi^t\nu_i(\zeta)\frac{d\bar{\zeta}}{d\zeta}=2\phi^t\frac{\mu_i(z,t)}{1-|\mu(z,t)|^2}\frac{\zeta_z}{\bar{\zeta}_{\bar z}}\frac{d\bar{\zeta}}{d\zeta},$$

where $\phi_i{}^t=\underline{\phi}_i{}^{t,0}\,d\zeta+\underline{\phi}_i{}^{t,1}d\bar{\zeta}$.

Next we consider the differentiability of $\phi_i{}^t$.

Set

$$\omega_i(h_j)=(\underline{\phi}_i{}^{t+h_j,0}(\zeta_j)_\zeta+\underline{\phi}_i{}^{t+h_j,1}(\overline{\zeta_j})_\zeta-\underline{\phi}_i{}^{t,0})d\zeta,$$

$$\sigma_i(h_j)=(\underline{\phi}_i{}^{t+h_j,0}(\zeta_j)_{\bar\zeta}+\underline{\phi}_i{}^{t+h_j,1}(\overline{\zeta_j})_{\bar\zeta}-\underline{\phi}_i{}^{t,1})d\bar{\zeta},$$

where $\phi_i{}^{t+h_j}=\underline{\phi}_i{}^{t+h_j,0}d\zeta_j+\underline{\phi}_i{}^{t+h_j,1}d\overline{\zeta_j}$.

Then $\phi_i{}^{t+h_j}\circ f_{t+h_j}\circ f_t{}^{-1}-\phi_i{}^t=\omega_i(h_j)+\sigma_i(h_j)$. We first show the following.

Lemma 2.

$$\lim_{h_j\to 0}\frac{\underline{\phi}_i{}^{t+h_j,0}(\zeta_j)_{\bar\zeta}}{h_j}=\underline{\phi}_i{}^{t,0}\frac{\mu_j(z,t)}{1-|\mu(z,t)|^2}\frac{\zeta_z}{\overline{\zeta_z}}$$

Proof. Remark that

$$\frac{\underline{\phi}_i{}^{t+h_j,0}(\zeta_j)_{\bar\zeta}}{h_j}=\underline{\phi}_i{}^{t+h_j,0}\frac{\nu(\zeta,h_j)}{h_j}(\zeta_j)_\zeta$$

$$=\underline{\phi}_i{}^{t+h_j,0}\frac{\mu_j(z,t)+\varepsilon_j(h_j)/h_j}{1-|\mu(z,t)|^2-\overline{\mu(z,t)}(h_j\mu_j(z,t)+\varepsilon_j(h_j))}\frac{\zeta_z}{\overline{\zeta_z}}(\zeta_j)_\zeta$$

This implies the conclusion.

Next we have

$$\underline{\phi}_i{}^{t+h_j,1}(\overline{\zeta_j})_{\bar\zeta}-\underline{\phi}_i{}^{t,1}$$

$$=(\underline{\phi}^{t+h_j}(\zeta_j)_\zeta-\underline{\phi}^t)\frac{\mu_i(z,t+h_j)}{1-|\mu(z,t+h_j)|^2}\frac{(\zeta_j)_z}{\overline{(\zeta_j)_z}}\frac{\overline{(\zeta_j)_\zeta}}{(\zeta_j)_\zeta}$$

$$+\underline{\phi}^t\left\{\frac{\mu_i(z,t+h_j)}{1-|\mu(z,t+h_j)|^2}-\frac{\mu_i(z,t)}{1-|\mu(z,t)|^2}\right\}\frac{(\zeta_j)_z}{\overline{(\zeta_j)_z}}\frac{\overline{(\zeta_j)_\zeta}}{(\zeta_j)_\zeta}$$

$$+\underline{\phi}^t\frac{\mu_i(z,t)}{1-|\mu(z,t)|^2}\left\{\frac{(\zeta_j)_z}{\overline{(\zeta_j)_z}}\frac{\overline{(\zeta_j)_\zeta}}{(\zeta_j)_\zeta}-\frac{\zeta_z}{\overline{\zeta_z}}\right\}.$$

For the third term, we remark that

$$\bar{z}_\zeta\zeta_z=\bar{z}_z-\bar{z}_{\bar\zeta}\bar{\zeta}_z=-\overline{\mu(z,t)}\,\overline{z_\zeta\zeta_z},$$

$$z_\zeta\zeta_z=z_z-z_{\bar\zeta}\bar{\zeta}_z=1-\overline{\bar{z}_\zeta\zeta_z\mu(z,t)}=1+|\mu(z,t)|^2z_\zeta\zeta_z,$$

hence $z_\zeta\zeta_z=\dfrac{1}{1-|\mu(z,t)|^2}$ and $\bar{z}_\zeta\zeta_z=\dfrac{-\overline{\mu(z,t)}}{1-|\mu(z,t)|^2}$.

We have

$$(\zeta_j)_\zeta \zeta_z = (\zeta_j)_z (z_\zeta \zeta_z + \mu(z, t+h_j)\bar{z}_\zeta \zeta_z)$$
$$= (\zeta_j)_z \left\{1 - \frac{\overline{\mu(z,t)}}{1-|\mu(z,t)|^2}(h_j \mu_j(z,t) + \varepsilon_j(h_j))\right\}.$$

It follows that

$$\frac{\overline{(\zeta_j)_\zeta \zeta_z}}{(\zeta_j)_z} = \frac{\overline{\zeta_z}}{\zeta_z}\left\{1 - \frac{\overline{\mu(z,t)}}{1-|\mu(z,t)|^2}(h_j \mu_j(z,t) + \varepsilon_j(h_j))\right\}$$

and

$$\frac{1}{h_j}\left\{\left(\overline{\frac{(\zeta_j)_\zeta \zeta_z}{(\zeta_j)_z}}\right) - \frac{(\zeta_j)_\zeta \zeta_z}{(\zeta_j)_z}\right\} - \frac{\overline{\mu(z,t)}\mu_j(z,t) - \mu(z,t)\overline{\mu_j(z,t)}}{1-|\mu(z,t)|^2}$$
$$= \frac{1}{h_j}(\varepsilon_j(h_j)\overline{\mu(z,t)} - \overline{\varepsilon_j(h_j)}\mu(z,t)).$$

Therefore

$$\lim_{h_j\to 0}\frac{1}{h_j}\left\{\frac{(\zeta_j)_z\overline{(\zeta_j)_\zeta}}{(\zeta_j)_z(\zeta_j)_\zeta} - \frac{\overline{\zeta_z}}{\zeta_z}\right\}$$
$$= \frac{1}{1-|\mu(z,t)|^2}\{\overline{\mu(z,t)}\mu_j(z,t) - \mu(z,t)\overline{\mu_j(z,t)}\}\frac{\overline{\zeta_z}}{\zeta_z}.$$

This convergence is given in the sense of the supremum norm. For the second term, we have

$$\frac{\mu_i(z,t+h_j)}{1-|\mu(z,t+h_j)|^2} - \frac{\mu_i(z,t)}{1-|\mu(z,t)|^2}$$
$$= \frac{h_j\mu_{ij}(z,t) + \varepsilon_{ij}(h_j) + \mu_i(z,t)|\mu(z,t+h_j)|^2 - \mu_i(z,t+h_j)|\mu(z,t)|^2}{(1-|\mu(z,t+h_j)|^2)(1-|\mu(z,t)|^2)}$$

and

$$|\mu(z,t+h_j)|^2 = |\mu(z,t)|^2 + h_j\{\mu_j(z,t)\overline{\mu(z,t)} + \overline{\mu_j(z,t)}\mu(z,t)\}$$
$$+ \{\varepsilon_j(h_j)\overline{\mu(z,t+h_j)} + \overline{\varepsilon_j(h_j)}\mu(z,t+h_j)\} + h_j^2|\mu_j(z,t)|^2 - |\varepsilon_j(h_j)|^2,$$

where $\varepsilon_{ij}(h_j) = \mu_i(z,t+h_j) - \mu_i(z,t) - h_j\mu_{ij}(z,t)$.

Hence

$$\lim_{h_j\to 0}\frac{1}{h_j}\left\{\frac{\mu_i(z,t+h_j)}{1-|\mu(z,t+h_j)|^2} - \frac{\mu_i(z,t)}{1-|\mu(z,t)|^2}\right\}$$
$$= \frac{\mu_{ij}(z,t)}{1-|\mu(z,t)|^2} + \frac{\mu_i(z,t)\{\mu_j(z,t)\overline{\mu(z,t)} + \mu(z,t)\overline{\mu_j(z,t)}\}}{(1-|\mu(z,t)|^2)^2}.$$

This convergence is also given in the sense of the supremum norm. Thus we have

Lemma 3.

$$\lim_{h_j\to 0}\frac{1}{h_j}(\phi_i^{t+h_j,1}(\overline{\zeta}_j)_{\bar{\zeta}} - \underline{\phi}_i^{t,1})$$
$$= \frac{1}{1-|\mu(z,t)|^2}\left\{\underline{\phi}_j^{t,0}\mu_i(z,t) + \underline{\phi}^t\mu_{ij}(z,t) + 2\underline{\phi}^t\frac{\mu_i(z,t)\mu_j(z,t)\overline{\mu(z,t)}}{1-|\mu(z,t)|^2}\right\}\frac{\overline{\zeta_z}}{\zeta_z}.$$

In a similar way we obtain the differentiability of $\phi_i^t$.

Theorem 2. There exists a differential $\phi_{ij}^t \in \Lambda_x(R^t) + \Lambda_{eo}(R^t)$ such that

$$\lim_{h_j\to 0}\frac{1}{h_j}\|\phi_i^{t+h_j}\circ f_{t+h_j}\circ f_t^{-1} - \phi_i^t - h_j\phi_{ij}^t\| = 0.$$

Proof. From Lemma 2 and 3, it follows that $\{\sigma_i(h_j)/h_j\}$ is a Cauchy sequence. The $\omega_i(h_j)+\sigma_i(h_j)=\phi_i^{t+h_j}\circ f_{t+h_j}\circ f_t^{-1}-\phi_i^t$ belongs to $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$ and $(\omega_i(h_j), \sigma_i(h_j))=0$. Hence we have

$$\begin{aligned}0&=<\omega_i(h_j)+\sigma_i(h_j),\ i^*(\omega_i(h_j)+\sigma_i(h_j))>\\&=<\omega_i(h_j)+\sigma_i(h_j),\ \omega_i(h_j)-\sigma_i(h_j)>\\&=\|\omega_i(h_j)\|^2-\|\sigma_i(h_j)\|^2.\end{aligned}$$

The $\{\omega_i(h_j)/h_j\}$ is also a Cauchy sequence.

Set $\lim_{h_j\to 0}\frac{\sigma_i(h_j)}{h_j}=\underline{\phi}_{ij}^{t,1}d\bar{\zeta}$, $\lim_{h_j\to 0}\frac{\omega_i(h_j)}{h_j}=\underline{\phi}_{ij}^{t,0}d\zeta$ and $\phi_{ij}^t=\underline{\phi}_{ij}^{t,0}d\zeta+\underline{\phi}_{ij}^{t,1}d\bar{\zeta}$. The $\phi_{ij}^t$ belongs to $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$ and satisfies the assertion.

Remark 2. From Lemma 2 and 3 we have

$$\begin{aligned}\underline{\phi}_{ij}^{t,1}&=\frac{1}{1-|\mu(z,t)|^2}\{\underline{\phi}_i^{t,0}\mu_j(z,t)+\underline{\phi}_j^{t,0}\mu_i(z,t)+\underline{\phi}^t\mu_{ij}(z,t)\}\frac{\zeta_z}{\bar{\zeta}_{\bar{z}}}\\&+\frac{2}{(1-|\mu(z,t)|^2)^2}\{\underline{\phi}^t\mu_i(z,t)\mu_j(z,t)\overline{\mu(z,t)}\}\frac{\zeta_z}{\bar{\zeta}_{\bar{z}}}.\end{aligned}$$

## 4. The First and Second Variational Formulas

Let a meromorphic differential $\psi^t=\underline{\psi}^t d\zeta$ on $R^t$ satisfy the same assumption as that of $\phi^t$. We know that various function theoretic quantities on $R^t$ are represented by $<\phi^t\circ f_t-\phi^0,\ \overline{\psi^0}>_R$.[3] We can obtain variational formulas of those quantities by the results in Section 3.

Theorem 3.

$$\begin{aligned}\frac{\partial}{\partial t_i}<\phi^t\circ f_t-\phi^0,\ \overline{\psi^0}>_R&=<\phi_i^t,\ \overline{\psi^t}>_{R^t}\\&=\mathrm{Re}\ i\iint_R\underline{\phi}^t\underline{\psi}^t\mu_i(z,t)\zeta_z^2dz\wedge d\bar{z}.\end{aligned}$$

Proof. By definition and Lemma 1.

$$\begin{aligned}&\frac{\partial}{\partial t_i}<\phi^t\circ f_t-\phi^0,\ \overline{\psi^0}>_R\\&=\lim_{h_i\to 0}\frac{1}{h_i}<\phi^{t+h_i}\circ f_{t+h_i}-\phi^t\circ f_t,\ \overline{\psi^0}>_R\\&=\lim_{h_i\to 0}<\frac{\phi^{t+h_i}\circ f_{t+h_i}\circ f_t^{-1}-\phi^t}{h_i},\ -*((*\overline{\psi^0})\circ f_t^{-1})>_{R^t}\\&=<\phi_i^t,\ -i^*(\overline{\psi^0}\circ f_t^{-1})>_{R^t}.\end{aligned}$$

Since $\phi_i^t$ and $\psi^t-\psi^0\circ f_t^{-1}$ belong to $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$,

$$\begin{aligned}<\phi_i^t,\ -i^*(\overline{\psi^0}\circ f_t^{-1})>_{R^t}&=<\phi_i^t,\ i^*(\overline{\psi^t-\psi^0\circ f_t^{-1}})-i^*\overline{\psi^t}>_{R^t}\\&=<\phi_i^t,\ \overline{\psi^t}>_{R^t}.\end{aligned}$$

Next, by Remark 1,

$$<\phi_i^t,\ \overline{\psi^t}>_{R^t}=<\underline{\phi}_i^{t,1}d\bar{\zeta},\ \overline{\psi^t}>_{R^t}=\mathrm{Re}\ i\iint_{R^t}\underline{\phi}^t\underline{\psi}^t\frac{\mu_i}{1-|\mu|^2}\frac{\zeta_z}{\bar{\zeta}_{\bar{z}}}d\zeta\wedge d\bar{\zeta}$$

$$=\operatorname{Re}\, i\iint_R \underline{\phi}^t \underline{\psi}^t \mu_i(z,t)\zeta_z^2 dz\wedge d\bar{z}.$$

In a similar way we can find the second variation.

Theorem 4.

$$\frac{\partial^2}{\partial t_j\partial t_i}<\phi^t\circ f_t-\phi^0,\ \overline{\psi^0}>_R=<\phi_{ij}^t,\ \overline{\psi^t}>_{R^t}$$

$$=\operatorname{Re}\, i\left\{\iint_R(\underline{\phi}_i^{t,0}\mu_j(z,t)+\underline{\phi}_j^{t,0}\mu_i(z,t))\underline{\psi}^t\zeta_z^2 dz\wedge d\bar{z}\right.$$

$$+\iint_R\underline{\phi}^t\underline{\psi}^t\mu_{ij}(z,t)\zeta_z^2 dz\wedge d\bar{z}$$

$$\left.+2\iint_R\underline{\phi}^t\underline{\psi}^t\frac{\mu_i(z,t)\mu_j(z,t)\overline{\mu(z,t)}}{1-|\mu(z,t)|^2}\zeta_z^2 dz\wedge d\bar{z}\right\}.$$

Proof. By Theorem 3 and Lemma 1,

$$\frac{\partial^2}{\partial t_j\partial t_i}<\phi^t\circ f_t-\phi^0,\ \overline{\psi^0}>_R$$

$$=\lim_{h_j\to 0}\frac{1}{h_j}\{<\phi_i^{t+h_j},\ \overline{\psi^{t+h_j}}>_{R^{t+h_j}}-<\phi_i^t,\ \overline{\psi^t}>_{R^t}\}$$

$$=\lim_{h_j\to 0}\left\{<\frac{\phi_i^{t+h_j}\circ f_{t+h_j}\circ f_t^{-1}-\phi_i^t}{h_j},\ -*((*\overline{\psi^{t+h_j}})\circ f_{t+h_j}\circ f_t^{-1})>_{R^t}\right.$$

$$\left.+<\phi_i^t,\ \frac{-i*(\overline{\psi^{t+h_j}}\circ f_{t+h_j}\circ f_t^{-1}-\overline{\psi^t})}{h_j}>_{R^t}\right\}$$

Since $-i*(\overline{\psi^{t+h_j}}\circ f_{t+h_j}\circ f_t^{-1}-\overline{\psi^t})\in i*\Lambda_x(R^t)+*\Lambda_{eo}(R^t)$, the second term vanishes. Therefore $\frac{\partial^2}{\partial t_j\partial t_i}<\phi^t\circ f_t-\phi^0,\ \overline{\psi^t}>_R=<\phi_{ij}^t,\ \overline{\psi^t}>_{R^t}$. By Bemark 2, we have

$$<\phi_{ij}^t,\ \overline{\psi^t}>_{R^t}=<\underline{\phi}_{ij}^{t,1}d\bar{\zeta},\ \overline{\psi^t}>_{R^t}$$

$$=\operatorname{Re}\, i\left\{\iint_{R^t}\frac{(\underline{\phi}_i^{t,0}\mu_j+\underline{\phi}_j^{t,0}\mu_i+\underline{\phi}^t\mu_{ij})}{1-|\mu|^2}\underline{\psi}^t\frac{\zeta_z}{\bar{\zeta}_z}d\zeta\wedge d\bar{\zeta}\right.$$

$$\left.+\iint_{R^t}\frac{2\underline{\phi}^t\mu_i\mu_j\bar{\mu}}{(1-|\mu|^2)^2}\underline{\psi}^t\frac{\zeta_z}{\bar{\zeta}_z}d\zeta\wedge d\bar{\zeta}\right\}.$$

From $d\zeta\wedge d\bar{\zeta}=(1-|\mu|^2)|\zeta_z|^2 dz\wedge d\bar{z}$, we obtain the conclusion.

Remark 3. For the first term of the variational formula, we have various representations. In fact

$$\operatorname{Re}\, i\iint_R\underline{\phi}_i^{t,0}\underline{\psi}^t\mu_j\zeta_z^2 dz\wedge d\bar{z}$$

$$=\operatorname{Re}\, i\iint_{R^t}\underline{\phi}_i^{t,0}\underline{\psi}^t\frac{\mu_j}{1-|\mu|^2}\frac{\zeta_z}{\bar{\zeta}_z}d\zeta\wedge d\bar{\zeta}=<\underline{\phi}_i^{t,0}d\zeta,\ i*(\overline{\underline{\psi}_j^{t,1}d\bar{\zeta}})>_{R^t}$$

$$=<\phi_i^t,\ \overline{\underline{\psi}_j^{t,1}d\bar{\zeta}}>_{R^t}=<\phi_i^t,\ i*(\overline{\psi_j^t-\underline{\psi}_j^{t,0}d\zeta})>_{R^t}$$

$$=<\phi_i^t,\ \overline{\underline{\psi}_j^{t,0}d\zeta}>_{R^t}=\frac{1}{2}<\phi_i^t,\ \overline{\psi_j^t}>,$$

and

$$\operatorname{Re}\, i\iint_R\underline{\psi}_j^{t,0}\underline{\phi}^t\mu_i\zeta_z^2 dz\wedge d\bar{z}=\frac{1}{2}<\psi_j^t,\ \overline{\phi_i^t}>=\frac{1}{2}<\phi_i^t,\ \overline{\psi_j^t}>.$$

The first term becomes $\mathrm{Re}\ i\left\{\iint_R (\phi_j^{t,0}\underline{\psi}^t+\underline{\psi}_j^{t,0}\underline{\phi}^t)\mu_i(z,t)\zeta_z{}^2 dz\wedge d\bar{z}\right.$ or $\frac{1}{2}\{<\phi_i,^t\ \overline{\psi_j{}^t}>+<\psi_i{}^t,\ \overline{\phi_j{}^t}>\}$.

Remark 4. For the third term, we have

$$2\iint_R \underline{\phi}^t\underline{\psi}^t\frac{\mu_i\mu_j\bar{\mu}}{1-|\mu|^2}\zeta_z{}^2 dz\wedge d\bar{z}=\iint_{R^t}(\underline{\phi}_j{}^{t,1}\underline{\psi}^t+\underline{\phi}^t\underline{\psi}_i{}^{t,1})\mu_i\bar{\mu}|\zeta_z|^2 dz\wedge d\bar{z}$$

and some other representations can be found using the same manner.

## 5. Recovering Deformations

As an example of a $C^2$-movement we will consider a deformation by recovering of local neighborhoods.

Let $\gamma$ be an analytic Jordan curve in $R$ and take local coordinates $\{\psi_k, z_k\}_{k=1,2}$ such that

$\psi_k(\gamma)=\{z_k;\ |z_k|=1\}$ and $\psi_2(p)=\dfrac{1}{\psi_1(p)}$ in an annulus which contains $\gamma$. Let $w_k=w_k(z_k, t)$ be a conformal mapping from $\{z_k:\ a\leqq|z_k|\leqq a^{-1}\}$ to an annulus $D_k$ such that

$$b_k<\inf_{|z_k|=a}|w_k(z_k,t)|<\inf_{|z_k|=a^{-1}}|w_k(z_k,t)|\leqq\sup_{|z_k|=a^{-1}}|w_k(z_k,t)|<\infty.$$

Denote $A_k(c_k, b_k)=\{z_k;\ c_k\leqq|z_k|\leqq b_k\}$, and $B_k(c_k, a^{-1})$ the annulus whose boundary is $\{w_k;\ |w_k|=c_k\}$ and $\{w_k(z_k);\ |z_k|=a^{-1}\}$. We can obtain a recovering surface $R^t$ as follow:

$$R^t=\left(R-\bigcup_{k=1}^{2}\psi_k{}^{-1}\{z_k:\ b_k\leqq|z_k|\leqq a^{-1}\}\right)\cup\bigcup_{k=1}^{2}B_k(c_k,\ a^{-1})$$

where $w_k(c_k<|w_k|<b_k)$ is identified with $p\in R$ if $\psi_k(p)=w_k$ and $w_1(z_1, t)$ $(a<|z_1|<a^{-1})$ is identified with $w_2\left(\dfrac{1}{z_1}, t\right)$. Now assume that there exists a quasiconformal mapping $\underline{f}_t$ from $A_k(c_k, a^{-1})$ to $B_k(c_k, a^{-1})$ such that

$$\underline{f}_t(z_k)=\begin{cases} w_k(z_k,\ t) & \text{on } A_k(1,\ a^{-1}) \\ z_k & \text{on } A_k(c_k,\ b_k)\end{cases}$$

and the Beltrami coefficient $\underline{\mu}(z_k, t)$ of $\underline{f}_t(z_k)$ satisfies the assumption in section 2. The $\underline{f}_t$ is extended to a quasiconformal mapping $f_t$ from $R$ to $R^t$ by setting an identity mapping on $R-\bigcup_{k=1}^{2}A_k(c_k, a^{-1})$. We can give some kind of $\underline{f}_t$. We recall the following fact.

Lemma 5.[6] Let $Q$ be analytic on $\{z:\ |z|=R\}$ and the Laurent expansion be $\sum_{n=-\infty}^{\infty}a_n z^n$. Let $H_Q$ be the harmonic function on $\{z:\ r<|z|<R\}$ such that

$$H_Q=\begin{cases} Q & \text{on } |z|=R \\ 0 & \text{on } |z|=r\end{cases}.$$

Then $H_Q$ has the following representation:

$$H_Q=b_0\log\frac{|z|}{r}+\sum_{n\neq 0} b_n z^n+\sum_{n\neq 0} c_n\bar{z}^n,$$

where $b_0=a_0/\log\frac{R}{r}$, for $n>0$

$$b_n=\frac{R^{2n}a_n}{R^{2n}-r^{2n}},\qquad b_{-n}=\frac{-r^{2n}a_{-n}}{R^{2n}-r^{2n}},$$

$$c_n=\frac{a_{-n}}{R^{2n}-r^{2n}},\qquad c_{-n}=\frac{-R^{2n}r^{2n}a_n}{R^{2n}-r^{2n}},$$

We have harmonic functions $H^1{}_k=H_{Q_1}+z_k$, $h^2{}_k=H_{Q_2}$ on $A_k(b_k,\ a)$ as in Lemma 5 for $Q_1(z_k)=w_k(z_k)-z_k$, $Q_2(z_k)=\log w_k(z_k)-\log z_k$. Set $H^2{}_k=z_k\exp h^2{}_k$ and

$$\underline{f}^i{}_k=\begin{cases} w_k(z_k) & \text{on } A_k(a,\ a^{-1})\\ H^i{}_k(z_k) & \text{on } A_k(b_k,\ a)\ .\\ z_k & \text{on } A_k(c_k,\ b_k)\end{cases}$$

When $\{w_k(z_k)\colon a<|z_k|<1\}$ contains a circle $\{w_k\colon |w_k|=d_k\}$ we have harmonic functions $H^3{}_k=H_{Q_3}+w_k$, $h^4{}_k=H_{Q_4}$ on $\{w_k\colon b_k<|z_k|<d_k\}$, where $Q_3(w_k)=z_k(w_k)-w_k$, $Q_4(w_k)=\log z_k(w_k)-\log w_k$. Set $H^4{}_k=w_k\exp h^4{}_k$,

$$\underline{g}^i{}_k=\begin{cases} z_k(w_k) & d_k<|w_k|\\ H^i{}_k(w_k) & b_k<|w_k|<d_k.\\ w_k & c_k<|w_k|<b_k\end{cases}$$

If the norms of the Beltrami coefficient of these $\underline{f}^i{}_k$, $\underline{g}^i{}_k$ are less than 1, these are univalent and become quasiconformal mappings. Set $\underline{f}^i{}_k=(\underline{g}^i{}_k)^{-1}$ $(i=3,\ 4)$. We will apply these to Theorem 3. Hereafter we often abreviate suffices as $z=z_k$, $H^i=H^i{}_k$.

For $i=1$, 2 we have

$$\mu_j w_z{}^2=H^i{}_{\bar{z}t_j}H^i{}_z-H^i{}_{\bar{z}}H^i{}_{zt_j}\ \text{on}\ A_k(b_k,\ a).$$

For $i=3$, 4 note that

$$w_z=\frac{\bar{z}_{\bar{w}}}{|z_w|^2-|z_{\bar{w}}|^2},\qquad w_{\bar{z}}=\frac{-z_{\bar{w}}}{|z_w|^2-|z_{\bar{w}}|^2},$$

$$w_{zt_j}=\frac{(\bar{z}_{\bar{w}})_{t_j}(|z_w|^2-|z_{\bar{w}}|^2)-\bar{z}_{\bar{w}}(|z_w|^2-|z_{\bar{w}}|^2)_{t_j}}{(|z_w|^2-|z_{\bar{w}}|^2)^2}$$

$$w_{\bar{z}t_j}=\frac{-(z_{\bar{w}})_{t_j}(|z_w|^2-|z_{\bar{w}}|^2)+z_{\bar{w}}(|z_w|^2-|z_{\bar{w}}|^2)_{t_j}}{(|z_w|^2-|z_{\bar{w}}|^2)^2}$$

and

$$\mu_j w_z{}^2=\frac{(\overline{H^i{}_{\bar{w}}})_{t_j}H^i{}_{\bar{w}}-(H^i{}_{\bar{w}})_{t_j}\overline{H^i{}_{\bar{w}}}}{(|H^i{}_w|^2-|H^i{}_{\bar{w}}|^2)^2},$$

where partial derivatives with respect to $t_j$ are those for functions with independent variables $z$ and $t_j$. Thus we have by theorem 3,

Theorem 5.

$$\frac{\partial}{\partial t_j}<\phi^t\circ f_t-\phi^0,\ \overline{\phi^0}>_R$$

$$=\mathrm{Re}\ \mathrm{i}\sum_{k=1}^{2}\iint_{B_k(b_k,\,a)}\underline{\phi}^t\underline{\psi}^t\frac{H^t_{\bar{z}t_j}H^t_z-H^t_{\bar{z}}H^t_{zt_j}}{|H^t_z|^2-|H^t_{\bar{z}}|^2}\mathrm{d}w\wedge\mathrm{d}\bar{w}\qquad i=1,\ 2$$

$$=\mathrm{Re}\ \mathrm{i}\sum_{k=1}^{2}\iint_{A_k(b_k,\,d_k)}\underline{\phi}^t\underline{\psi}^t\frac{(\overline{H}^t_{\bar{w}})_{t_j}H^t_{\bar{w}}-(H^t_{\bar{w}})_{t_j}\overline{H}^t_{\bar{w}}}{|H^t_w|^2-|H^t_{\bar{w}}|^2}\mathrm{d}w\wedge\mathrm{d}\bar{w}\qquad i=3,\ 4.$$

Remark 4. In a bordered surface, when $\{z_1:\ |z_1|=a^{-1}\}$ corresponds to the border, this gives a boundary variation if we take $b_2=c_2=a$.

Remark 5. When $\gamma$ is homologous 0, this gives a Schiffer's interior variation.[1]

At last we consider a pinching deformation. When $w_k=(1+s_k(t))z_k$, Theorem 5 represents a pinching deformation and the formula for $i=2$ is useful.[4] In fact we have

$$Q=\log(1+s(t)),\ h=A(t)\log\frac{|z|}{b},\ H=z\left(\frac{|z|}{b}\right)^{A(t)},$$

where $A(t)=\log(1+s(t))/\log\frac{a}{b}$. It follows that

$$H_z=\left(\frac{|z|}{b}\right)^{A(t)}\left(1+\frac{A(t)}{2}\right),\qquad H_{\bar{z}}=\frac{A(t)}{2}\left(\frac{|z|}{b}\right)^{A(t)}\frac{z}{\bar{z}},$$

$$H_{\bar{z}t}H_z-H_{zt}H_{\bar{z}}=\frac{A'(t)}{2}\left(\frac{|z|}{b}\right)^{2A(t)}\frac{z}{\bar{z}},$$

$$A'(t)=s'(t)/(1+s(t))\log\frac{a}{b},\ \ \frac{z}{\bar{z}}=\left(\frac{|z|}{b}\right)^{-2\mathrm{i}\,\mathrm{Im}A(t)}\frac{w}{\bar{w}},$$

$$|H_z|^2-|H_{\bar{z}}|^2=\frac{1}{4}\left(\frac{|z|}{b}\right)^{2\mathrm{Re}A(t)}(|2+A(t)|^2-|A(t)|^2)$$

$$=\left(\frac{|z|}{b}\right)^{2\mathrm{Re}A(t)}E(t)/\log\frac{a}{b},$$

where $E(t)=\log\frac{a}{b}+\log|1+s(t)|$.

Hence

$$\frac{H_{\bar{z}t}H_z-H_{zt}H_{\bar{z}}}{|H_z|^2-|H_{\bar{z}}|^2}=\frac{s'(t)}{2(1+s(t))E(t)}\frac{w}{\bar{w}}.$$

For $\underline{\phi}^t=\sum a_nw^n,\ \underline{\psi}^t=\sum b_mw^m$

$$\mathrm{i}\iint_{A(b,a)}\underline{\phi}^t\underline{\psi}^t\mu_iH_z^2\mathrm{d}z\wedge\mathrm{d}\bar{z}$$

$$=\mathrm{i}\frac{s'(t)}{2(1+s(t))E(t)}\iint_{A(b,\,|1+s(t)|a)}\sum a_nb_mw^{n+m}\frac{w}{\bar{w}}\mathrm{d}w\wedge\mathrm{d}\bar{w}$$

$$=\frac{s'(t)}{(1+s(t))E(t)}\int_b^{|1+s(t)|a}\int_0^{2\pi}\sum a_nb_mr^{n+m+1}e^{\mathrm{i}(n+m+2)\theta}\mathrm{d}r\mathrm{d}\theta$$

$$=\frac{2\pi s'(t)}{(1+s(t))}\sum_{n+m=-2}a_nb_m.$$

Therefore we have

Theorem 6.

$$\frac{\partial}{\partial t_i}<\phi^t\circ f_t-\phi^0,\ \overline{\psi^0}>_R$$

$$=\mathrm{Re}\ 2\pi\sum_{k=1}^{2}\frac{1}{1+s_k(t)}\frac{\partial}{\partial t_i}s_k(t)\sum_{n+m=-2}a_{n,k}(t)b_{m,k}(t).$$

Set $s_3(t)=(1+s_1(t))(1+s_2(t))$. Then $w_2=\frac{s_3(t)}{w_1}$ and

$$\sum a_{n,2}(t)w_2^n dw_2=-\sum a_{n,2}(t)s_3(t)^{n+1}w_1^{-n-2}dw_1.$$

Hence we have

$$a_{n,2}(t)=-a_{-n-2,1}(t)s_3(t)^{-n-1},\ (\text{resp. } b_{n,2}(t)=-b_{-n-2,1}(t)s_3(t)^{-n-1}).$$

We can often assume that the Dirichlet norms of $\phi^t$ and $\psi^t$ in $A_k(b_k, a)$ are uniformly bounded with respect to $t$. Since

$$\|\phi^t\|^2_{A_k(b_k,a)}=2\pi\sum|a_{n,k}(t)|^2(a^{2n+2}-(b_k)^{2n+2})/(n+1),$$

$|a_{n,k}(t)|$ (resp. $|b_{n,k}(t)|$) are also uniformly bounded with respect to $t$. If $s_3(t)$ tends to infinity, then $a_{n,k}(t)$ (resp. $b_{n,k}(t)$) converges to 0 for $n\geq 0$.

For the second variation we have

$$\mu=\frac{H_{\bar z}}{H_z}=\frac{A(t)}{2+A(t)}\frac{z}{\bar z},$$

$$\mu_i=\frac{2A_i(t)}{(2+A(t))^2}\frac{z}{\bar z}\qquad\left(A_i(t)=\frac{\partial A(t)}{\partial t_i}\right),$$

$$\mu_{ij}=\left\{\frac{2A_{ij}(t)}{(2+A(t))^2}-\frac{4A_i(t)A_j(t)}{(2+A(t))^3}\right\}\frac{z}{\bar z}\qquad\left(A_{ij}(t)=\frac{\partial^2 A(t)}{\partial t_j\partial t_i}\right),$$

$$\frac{\mu_{ij}H_z^2}{|H_z|^2-|H_{\bar z}|^2}=\frac{2A_{ij}(2+A(t))-4A_i(t)A_j(t)}{(|2+A(t)|^2-|A(t)|^2)(2+A(t))}\frac{w}{\bar w}\left(=\frac{X(t)}{4E(t)}\frac{w}{\bar w}\right),$$

$$\frac{\mu_i\mu_j\bar\mu H_z^2|H_z|^2}{(|H_z|^2-|H_{\bar z}|^2)^2}=\frac{4A_i(t)A_j(t)\overline{A(t)}}{(|2+A(t)|^2-|A(t)|^2)^2(2+A(t))}\frac{w}{\bar w}\left(=\frac{Y(t)}{4E(t)}\frac{w}{\bar w}\right).$$

It follows that

$$i\iint\underline{\phi^t\psi^t}\left(\mu_{ij}+\frac{2\mu_i\mu_j\bar\mu}{1-|\mu|^2}\right)H_z^2dz\wedge d\bar z$$
$$=\pi(X(t)+2Y(t))\sum_{n+m=-2}a_n(t)b_m(t),$$

where $X(t)=2\log\frac{a}{b}\left(A_{ij}(t)-\frac{2A_i(t)A_j(t)}{2+A(t)}\right)$,

$$Y(t)=\left(\log\frac{a}{b}\right)^2 A_i(t)A_j(t)\overline{A(t)}/(2+A(t))E(t).$$

From $A_i=s_i/(1+s)\log\frac{a}{b}$, $A_{ij}=(s_{ij}(1+s)-s_is_j)/(1+s)^2\log\frac{a}{b}$ we have

$$X+2Y=2\log\frac{a}{b}\left\{A_{ij}+A_iA_j\left(-2E+\bar A\log\frac{a}{b}\right)/E(2+A)\right\}$$
$$=2\log\frac{a}{b}\left\{A_{ij}-A_iA_j\log\frac{a}{b}/E\right\}$$
$$=2\{s_{ij}(1+s)-s_is_j(E+1)/E\}/(1+s)^2.$$

Therefore

$$\frac{1}{2}\{<\phi_i^t,\overline{\psi_j^t}>+<\psi_i^t,\overline{\phi_j^t}>\}$$
$$=\frac{\partial}{\partial t_j}\left(2\pi\mathrm{Re}\sum_{k=1}^{2}\frac{1}{1+s_k(t)}\frac{\partial}{\partial t_i}s_k(t)\sum_{n+m=-2}a_{n,k}(t)b_{m,k}(t)\right)$$
$$-2\pi\mathrm{Re}\sum_{k=1}^{2}\frac{1}{1+s_k(t)}\left\{\frac{\partial^2 s_k(t)}{\partial t_j\partial t_i}-\frac{1}{1+s_k(t)}\frac{\partial s_k(t)}{\partial t_i}\frac{\partial s_k(t)}{\partial t_j}\frac{E(t)+1}{E(t)}\right\}\sum_{n+m=-2}a_{n,k}(t)b_{m,k}(t)).$$

Thus we have

Thorem 7.

$$\frac{1}{2}\{<\phi_i^t, \overline{\psi_j^t}>+<\psi_i^t, \overline{\phi_j^t}>\}$$

$$=2\pi \mathrm{Re} \sum_{k=1}^{2} \frac{1}{1+s_k(t)} \frac{\partial s_k(t)}{\partial t_i}$$

$$\times \left\{\frac{\partial}{\partial t_j} \sum_{n+m=-2} a_{n,k}(t) b_{m,k}(t) + \frac{1}{(1+s_k(t))E(t)} \frac{\partial s_k(t)}{\partial t_j} \sum_{n+m=-2} a_{n,k}(t) b_{m,k}(t)\right\}.$$

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## References

1) F. Gardiner, *Duke Math. J.*, **42**, 371–380 (1975)
2) Y. Kusunoki and F. Maitani, *Math. Z.*, **181**, 435–450 (1982).
3) F. Maitani, *J. Math. Kyoto Univ.*, **24**, 49–66 (1984).
4) M. Taniguchi, *J. Math. Kyoto Univ.*, **29**, 45–56 (1989).
5) M. Taniguchi, *Kodai Math. J.*, **12**, 283–295 (1989).
6) A. Yamada, *Kodai Math. J.*, **3**, 114–143 (1980).