Quasiconformal Deformations of an Arbitrary Riemann Surface and Variational Formulas

By

Fumio MAITANI

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Abstract

The purpose of this paper is to give variational formulas for Riemann's period matrices and certain kinds of meromorphic differentials on an arbitrary open Riemann surface which is deformed by quasiconformal homeomorphisms depending on a complex parameter. As the quasiconformal deformation we consider Riemann surfaces with conformal structures decided by Beltrami differentials depending holomorphically on a complex parameter. For the sake of discussion on general open Riemann surfaces we introduce a notion of behavior spaces in the Hilbert space of first order differential forms. The mapping induced from a quasiconformal homeomorphism preserves the behavior space. Our variational formulas are valid for the class of meromorphic differentials restricted by the behavior space. We shall show examples that each element of our period matrix is holomorphic if branch points and boundary curves vary holomorphically on a covering surface of the complex plane.

1. Introduction

We are concerned with the dependence of the fundamental quantity on a Riemann surface as it varies with a parameter. As is well known, on the Teichmüller space of compact Riemann surfaces of genus $g>1$, Ahlfors introduced a complex analytic structure in which all the elements of the Riemann matrix are holomorphic functions and showed that it is uniquely determined[1]. Recently Kusunoki discussed this for the case of non compact Riemann surfaces belonging to class O'' and showed that they are holomorphic with respect to the Bers coordinate in the Teichmüller space of Riemann surfaces of class O''[6]. On the other hand, Shiba formulated some theorems on arbitrary open Riemann surfaces by using behavior spaces on the real number field[14].

In this paper we study the above problem on arbitrary Riemann surfaces by means of behavior spaces on the complex number field[9]. We consider Beltrami differentials with a complex parameter on a Riemann surface and a family of Riemann surfaces with the complex structure induced by the Beltrami differentials. We choose normal meromorphic differentials which are defined from our behavior spaces. Then, roughly speaking, each element of the period matrix for our normal meromorphic differentials is holomorphic if the Beltrami differentials are holomorphic for the parameter. For this purpose we shall give some variational formulas with respect to these normal meromorphic differentials. As examples, we can show that each element of our period matrix is holomorphic if branch points or boundary curves vary holomorphically on a covering surface of the complex plane. This paper is rewritten using more general behavior spaces in 9) instead of the normal behavior spaces in the jont paper[7] with Pf. Kusunoki. Note that the sign of intersection number follows 3) and 9) but is different from the one in 7).

2. Behavior Spaces

Let R be a Riemann surface, $\{G_n\}$ be a canonical exhaustion of R and $\Xi=\{A_i, B_i\}$ be a canonical homology basis modulo dividing cycles associated with $\{G_n\}$ such that (i) $A_i\cap B_i$ consists of a point, (ii) $(A_i\cup B_i)\cap(A_j\cup B_j)=\phi$ for $i\neq j$, (iii) The intersection numbers satisfy $A_i\times A_j=B_i\times B_j=0$, $A_i\times B_j=0$ for $i\neq j$ and $A_i\times B_i=1$, where A_i crosses B_i from right to left. Let $\Gamma=\Gamma(R)$ be a Hilbert space whose elements are complex differentials on R and whose inner product is given by the form:

$$(\omega_1, \omega_2)=\iint_R \omega_1\wedge\overline{\omega_2^*}=i\iint_R(a_1\bar{a}_2+b_1\bar{b}_2)dzd\bar{z},$$

where $\omega_j=a_jdz+b_jd\bar{z}(j=1, 2)$ in terms of a local parameter z. As for the notations of subspaces in Γ we follow Ahlfors-Sario[3], for instance, Γ_c, Γ_h, Γ_{hse} and Γ_{eo} denote the space of closed, harmonic, harmonic semiexact differentials and the space of differentials of Dirichlet potentials[5].

We use the following subspace Γ_x or Γ_h in this paper.

Definition. For a sequence of real numbers $\{a_i, b_i\}$ $(a_i\neq 0)$, *we call a subspace* Γ_x *of* Γ_h (a_i, b_i)*-behavior space if* Γ_x *satisfies (i)* $\Gamma_x\subset\Gamma_{hse}$, *(ii)* $\Gamma_x\dotplus\Gamma_x^*=\Gamma_h$, *(iii)* $\Gamma_x=\overline{\Gamma_x}$, *(iv)* $a_i\int_{A_i}\omega=b_i\int_{B_i}\omega$ *for any i and* $\omega\in\Gamma_x$.

We know the following.

Proposition 1. (cf. 9)) On an arbitrary Riemann surface, there exists an (a_i, b_i)*-behavior space for any sequence of real numbers* $\{a_i, b_i\}$ $(a_i\neq 0)$.

Now we have

Lemma 1. Let Γ_x *be an* (a_i, b_i)*-behavior space. Then*

$$\lim_{n\to\infty}\int_{\partial G_n} W_1\bar{\omega}_2=0 \text{ for any } \omega_1, \omega_2\in\Gamma_x,$$

where W_1 *is a primitive function of* ω_1 *on* $R-\cup(A_i\cup B_i)$.

Proof. From conditions (i), (iv) of Γ_x, we have

$$\begin{aligned}(\omega_1, \omega_2^*)&=\lim_{n\to\infty}\left\{-\int_{\partial G_n}W_1\bar{\omega}_2+\sum_{G_n}\left[\int_{A_i}\omega_1\int_{B_i}\bar{\omega}_2-\int_{B_i}\omega_1\int_{A_i}\bar{\omega}_2\right]\right\}\\&=-\lim_{n\to\infty}\int_{\partial G_n}W_1\bar{\omega}_2.\end{aligned}$$

On the other hand, by condition (ii) $(\omega_1, \omega_2^*)=0$. Thus the conclusion follows.

Similarly we have

Lemma 2. $\lim_{n\to\infty}\int_{\partial G_n} f\bar{\sigma}=0$ *for* $df\in\Gamma_{e0}^1$, $\sigma\in\Gamma_c^1$.

$\lim_{n\to\infty}\int_{\partial G_n} S\bar{\sigma}_0=0$ *for* $dS\in\Gamma_{se}^1$, $\sigma_0\in\Gamma_{e0}^1$.

We remark that

Lemma 3. Let a harmonic differential ω *satisfy* $a_i\int_{A_i}\omega=b_i\int_{B_i}\omega$ *for any* i. *If* ω *is equal to a differential* σ *in* $\Gamma_x+\Gamma_{e0}$ *in a neighbourhood of the ideal boundary, then* $\omega\in\Gamma_x$.

Proof. It is clear that $\omega\in\Gamma_h$ and for any $\omega_1\in\Gamma_x$.

$$(\omega_1,\omega^*)=\lim_{n\to\infty}\left\{-\int_{\partial G_n}W_1\bar{\omega}+\sum_{G_n}\left[\int_{A_i}\omega_1\int_{B_i}\bar{\omega}-\int_{B_i}\omega_1\int_{A_i}\bar{\omega}\right]\right\}$$
$$=\lim_{n\to\infty}-\int_{\partial G_n}W_1\bar{\omega}=0.$$

Hence ω is orthogonal to Γ_x^* and $\omega\in\Gamma_x$.

Here we define a boundary behavior of a meromorphic differential.

Definition. For an (a_i, b_i)*-behavior space* Γ_x, *a meromorphic differential* ψ *has* Γ_x*-behavior if there exist a neighbourhood* V *of the ideal boundary and differentials* $\omega\in\Gamma_x$, $\omega_0\in\Gamma_{e0}$ *such that* $\psi=\omega+\omega_0$ *on* V.

By Lemmas 1 and 2, we have

Proposition 2. (*cf.* 9)) *Let meromorphic differentials* ψ_1 *and* ψ_2 *have* Γ_x*-behavior. Then*

$$\lim_{n\to\infty}\int_{\partial G_n}\Psi_1\bar{\psi}_2=0,\quad \lim_{n\to\infty}\int_{\partial G_n}\Psi_1\psi_2=0,$$

where Ψ_1 *is a primitive function of* ψ_1 *on* $R-\cup(A_i\cup B_i)$.

We know the existence of the following elementary meromorphic differentials with Γ_x-behavior.

Proposition 3. (*cf.* 9)) *Let* Γ_x *be an* (a_i, b_i)*-behavior space on* R. *Then there exist meromorphic differentials with* Γ_x*-behavior* $\psi_{j,x}$, $\tilde{\psi}_{j,x}$, $\psi_{p,n,x}$ *and* $\psi_{p,q,x}$ *such that*

(*i*) $\psi_{j,x}$ *is holomorphic and* $a_i\int_{A_i}\psi_{j,x}=b_i\int_{B_i}\psi_{j,x}-a_i\delta_{ij}$,

(*ii*) $\tilde{\psi}_{j,x}$ *is holomorphic and* $a_i\int_{A_i}\tilde{\psi}_{j,x}=b_i\int_{B_i}\tilde{\psi}_{j,x}-b_i\delta_{ij}$,

(*iii*) $\psi_{p,n,x}$ *has the singularity* $-d(1/z^n)$ *only at* p (z *is a fixed local parameter about* p *and* $n\geq 1$), *and satisfies* $a_i\int_{A_i}\psi_{p,n,x}=b_i\int_{B_i}\psi_{p,n,x}$,

(*iv*) $\psi_{p,q,x}$ *has the singularities* $\frac{dz}{z}$ *at* p *and* $-\frac{dw}{w}$ *at* q (w *is a fixed local parameter about* q) *and is regular analytic elsewhere. Further it satisfies* $a_i\int_{A_i}\psi_{p,q,x}=b_i\int_{B_i}\psi_{p,q,x}$,

where $\delta_{ij}=0$ *for* $i\neq j$, $=1$ *for* $i=j$,

In the classical theory there are some relations between the normal integrals. In our case we have corresponding relations and show them by the calculation of inner products between these elementary differentials.

Proposition 4. Let us set $\int_{B_i}\psi_{j,x}=T_{ij}=T_{ij}'+iT_{ij}''(T_{ij}', T_{ij}''$ *are real). Then the matrix* $(T_{ij})_{i\leq k, j\leq k}$ $(0<k\leq$ *genus of* $R)$ *is symmetric and* $(T_{ij}'')_{i\leq k, j\leq k}$ *is positive definite.*

Proof. Since $\psi_{i,x}$ and $\psi_{j,x}$ have Γ_x-behavior and the normalization, we have

$$\begin{aligned}0&=(\psi_{i,x}, \overline{\psi_{j,x}^*})\\&=\lim_{n\to\infty}\left\{-\int_{\partial G_n}\Psi_{i,x}\psi_{j,x}+\sum_{G_n}\left[\int_{A_l}\psi_{i,x}\int_{B_l}\psi_{j,x}-\int_{B_l}\psi_{i,x}\int_{A_l}\psi_{j,x}\right]\right\}\\&=\int_{B_j}\psi_{i,x}-\int_{B_i}\psi_{j,x}=T_{ji}-T_{ij}.\end{aligned}$$

Hence (T_{ij}) is symmetric. If we set $\omega=\sum_{i=1}^{k} c_i\psi_{i,x}$, then

$$\begin{aligned}0\leq\|\omega\|^2&=\sum_{i,j}^{k}c_i\bar{c}_j(\psi_{i,x}, i\psi_{j,x}^*)\\&=-i\sum_{i,j}^{k}c_i\bar{c}_j\lim_{n\to\infty}\left\{-\int_{\partial G_n}\Psi_{i,x}\overline{\psi_{j,x}}+\sum_{G_n}\left[\int_{A_l}\psi_{i,x}\int_{B_l}\overline{\psi_{j,x}}-\int_{B_l}\psi_{i,x}\int_{A_l}\overline{\psi_{j,x}}\right]\right\}\\&=-i\sum_{i,j}^{k}c_i\bar{c}_j\left(\int_{B_j}\psi_{i,x}-\int_{B_i}\overline{\psi_{j,x}}\right)\\&=-i\sum_{i,j}^{k}c_i\bar{c}_j(T_{ji}-\bar{T}_{ij})=2\sum_{i,j}^{k}c_i\bar{c}_jT_{ij}''.\end{aligned}$$

Thus T_{ij}'' is positive definite.

We set

$$\Psi_j^x(p', q')=\int_{q'}^{p'}\psi_{j,x},$$

$$\Psi_{p,n}^x(p', q')=\int_{q'}^{p'}\psi_{p,n,x},$$

$$\Psi_{p,q}^x(p', q')=\int_{q'}^{p'}\psi_{p,q,x},$$

where p, q, p', $q'\in R-\cup(A_i\cup B_i)$ and the paths of these integrals are taken in $R-\cup(A_i\cup B_i)$. Then $\Psi_j^x(p', q')$, $\Psi_{p,n}^x(p', q')$ and $\Psi_{p,q}^x(p', q')$ are meromorphic functions with variable p' in $R-\cup(A_i\cup B_i)$. We can put the relations among these functions in order as

Proposition 5.

(*i*) $\int_{B_j}\psi_{p,q,x}=2\pi i\Psi_j^x(p, q)$,

(*ii*) $\int_{B_j}\psi_{p,n,x}=\frac{2\pi i}{(n-1)!}\frac{d^n}{dz^n}\Psi_j^x(z,\cdot),|_{z=0}$

(*iii*) $\Psi_{p,q}^x(p', q')=\Psi_{p',q'}^x(p, q)$,

(*iv*) $\frac{1}{(m-1)!}\frac{d^m}{dw^m}\Psi_{p,n}^x(w,\cdot)|_{w=0}=\frac{1}{(n-1)!}\frac{d^n}{dz^n}\Psi_{q,m}^x(z,\cdot)|_{z=0}$,

(*v*) $\Psi_{p,n}^x(p', q')=\frac{1}{(n-1)!}\frac{d^n}{dz^n}\Psi_{p',q'}^x(z,\cdot)|_{z=0}$, *where z and w are the local parameters surrounding p and q which define the singularities of* $\psi_{p,n,z}$ *and* $\psi_{p,q,z}$.

Proof. Let V_p, V_q, $V_{p'}$ and $V_{q'}$ be disjoint parametric disks surrounding p, q, p' and q' respectively, which do not intersect $\cup(A_i\cup B_i)$. Then we have the following by Lemmas 1, 2 and the period normalization.

$$\text{(i)}\quad 0=(\psi_{j,z}, \overline{\psi_{p,q,z}{}^*})_{R-(V_p\cup V_q)}$$
$$=\lim_{n\to\infty} -\int_{\partial(G_n-V_p\cup V_q)} \Psi_j{}^z\psi_{p,q,z}+\sum_{G_n}\left[\int_{A_i}\psi_{j,z}\int_{B_i}\psi_{p,q,z}-\int_{B_i}\psi_{j,z}\int_{A_i}\psi_{p,q,z}\right]$$
$$=2\pi i\Psi_j{}^z(p,q)-\int_{B_j}\psi_{p,q,z},$$

$$\text{(ii)}\quad 0=(\psi_{j,z}, \overline{\psi_{p,n,z}{}^*})_{R-V_p}$$
$$=\int_{\partial V_p}\Psi_j{}^z\psi_{p,n,z}-\int_{B_j}\psi_{p,n,z}$$
$$=\frac{2\pi i}{(n-1)!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi_j{}^z(z,\cdot)|_{z=0}-\int_{B_j}\psi_{p,n,z}$$

$$\text{(iii)}\quad 0=(\psi_{p,q,z}, \overline{\psi_{p',q',z}{}^*})_{R-V_p\cup V_q\cup V_{p'}\cup V_{q'}}$$
$$=\int_{\partial(V_p\cup V_q)}\Psi^z_{p,q}\psi_{p',q',z}+\int_{\partial(V_{p'}\cup V_{q'})}\Psi^z_{p,q}\psi_{p',q',z}$$
$$=2\pi i\Psi^z_{p,q}(p', q')+\int_{\partial(V_{p'}\cup V_{q'})}\{\mathrm{d}(\Psi^z_{p,q}\Psi^z_{p',q'})-\Psi^z_{p',q'}\psi_{p,q,z}\}$$
$$=2\pi i\Psi^z_{p,q}(p', q')-2\pi i\Psi^z_{p',q'}(p, q),$$

$$\text{(iv)}\quad 0=(\psi_{p,n,z}, \overline{\psi_{q,m,z}{}^*})_{R-V_p\cup V_q}$$
$$=\int_{\partial(V_p\cup V_q)}\Psi^z_{p,n}\psi_{q,m,z}$$
$$=\int_{\partial V_q}\Psi^z_{p,n}\psi_{q,m,z}+\int_{\partial V_p}\{\mathrm{d}(\Psi^z_{p,n}\Psi^z_{q,m})-\Psi^z_{q,m}\psi_{p,n,z}\}$$
$$=\frac{2\pi i}{(m-1)!}\frac{\mathrm{d}^m}{\mathrm{d}w^m}\Psi^z_{p,n}(w,\cdot)|_{w=0}-\frac{2\pi i}{(n-1)!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi^z_{q,m}(z,\cdot)|_{z=0},$$

$$\text{(v)}\quad 0=(\psi_{p,n,z}, \overline{\psi_{p',q',z}{}^*})_{R-V_p\cup V_q\cup V_{p'}\cup V_{q'}}$$
$$=\int_{\partial(V_p\cup V_q)}\Psi^z_{p,n}\psi_{p',q',z}+\int_{\partial(V_{p'}\cup V_{q'})}d(\Psi^z_{p,n}\Psi^z_{p',q'})-\Psi^z_{p',q'}\psi_{p,n,z}$$
$$=2\pi i\Psi^z_{p,n}(p', q')-\frac{2\pi i}{(n-1)!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi^z_{p',q'}(z,\cdot)|_{z=0}.$$

Q.E.D..

3. Deformations of behavior spaces

We consider a Beltrami differential $\mu(z)\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}$ ($\|\mu\|_\infty=\operatorname{esssup}_z|\mu|<1$) on R and denote by R_μ the Riemann surface whose conformal structure is given by $\mathrm{d}s=|\mathrm{d}z+\mu\mathrm{d}\bar{z}|$ in terms of a local parameter z on R. Let f be the quasiconformal homeomorphisms from R to R_μ whose Beltrami differential is $\mu\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}$, i.e.

$$\frac{\zeta_{\bar{z}}}{\zeta_z}=\frac{(\Pi_\mu\circ f\circ\Pi^{-1})_{\bar{z}}}{(\Pi_\mu\circ f\circ\Pi^{-1})_z}=\mu(z),$$

where Π and Π_μ are local homeomorphisms from R and R_μ to the complex planes z and ζ respectively. Then f induces an isomorphism $f^\sharp$ from $\Gamma(R)$ to $\Gamma(R_\mu)$:

$$f^\sharp(\omega)=[A(\Pi\circ f^{-1})(\Pi\circ f^{-1})_\zeta+B(\Pi\circ f^{-1})(\overline{\Pi\circ f^{-1}})_\zeta]d\zeta \\ +[A(\Pi\circ f^{-1})(\Pi\circ f^{-1})_{\bar\zeta}+B(\Pi\circ f^{-1})(\overline{\Pi\circ f^{-1}})_{\bar\zeta}]d\bar\zeta$$

where $\omega=A(z)dz+B(z)d\bar z$ in terms of a local parameter z in a neighbourhood of p, ζ is a local parameter about $p'=f(p)$ and $(\Pi\circ f^{-1})_\zeta$, $(\Pi\circ f^{-1})_{\bar\zeta}$, $(\overline{\Pi\circ f^{-1}})_\zeta$, $(\overline{\Pi\circ f^{-1}})_{\bar\zeta}$ are distributional derivatives of $(\Pi\circ f^{-1})$ and $(\overline{\Pi\circ f^{-1}})$ respectively. Let P_h denote the projection from Γ to Γ_h and by $f_h^\sharp$ the composite mapping $P_h\circ f^\sharp$. Similarly for the inverse mapping f^{-1} we can define $(f^{-1})^\sharp$ and $(f^{-1})_h^\sharp$.

We know

Lemma 4. (*cf.* 10), 11)) *The mappings* $(f^{-1})^\sharp\circ f^\sharp, f^\sharp\circ(f^{-1})^\sharp$ *and* $(f^{-1})_h^\sharp\circ f_h^\sharp, f_h^\sharp\circ(f^{-1})_h^\sharp$ *are identity mappings of* Γ *and* Γ_h *respectively. Further* $f^\sharp$ (*resp.* $f_h^\sharp$) *gives an isomorphism between* $\Gamma(R)$ *and* $\Gamma(R_\mu)$ (*resp.* $\Gamma_h(R)$ *and* $\Gamma_h(R_\mu)$). *If* $\|\mu\|_\infty\le k<1$, *then*

$$\|f^\sharp(\tau)\|^2_{R_\mu}\le\frac{1+k}{1-k}\|\tau\|^2_R \quad \text{for } \tau\in\Gamma(R)$$

$$\|f_h^\sharp(\omega)\|^2_{R_\mu}\le\frac{1+k}{1-k}\|\omega\|^2_R \quad \text{for } \omega\in\Gamma_h(R).$$

Let $\sigma(C)^*\in\Gamma_{ho}(R)^*$ be the period reproducing differential for a cycle C on R and $\sigma(f(C))^*\in\Gamma_{ho}(R_\mu)^*$ be the period reproducing differential for a cycle $f(C)$ on R_μ. We also know

Lemma 5. (*cf.* 10), 11))

$$(f^\sharp(\tau),\ \sigma(f(C))^*)_{R_\mu}=(\tau,\ \sigma(C)^*)_R \quad \text{for any } \tau\in\Gamma_c(R),$$
$$(f_h^\sharp(\omega),\ \sigma(f(C))^*)_{R_\mu}=(\omega,\ \sigma(C)^*)_R \quad \text{for any } \omega\in\Gamma_h(R),$$
$$f^\sharp(\Gamma_y(R))=\Gamma_y(R_\mu), \quad \text{where } \Gamma_y=\Gamma_c,\ \Gamma_{se},\ \Gamma_e \text{ and } \Gamma_{eo},$$
$$f_h^\sharp(\Gamma_z(R))=\Gamma_z(R_\mu), \quad \text{where } \Gamma_z=\Gamma_{hse},\ \Gamma_{he}\ \Gamma_{ho} \text{ and } \Gamma_{hm}.$$

We remark that

Proposition 6.

$$(f^\sharp(\tau_1)^*,\ f^\sharp(\tau_2^*))_{R_\mu}=(\tau_1,\ \tau_2)_R \quad \text{for any } \tau_1,\ \tau_2\in\Gamma(R),$$
$$(f_h^\sharp(\omega_1)^*,\ f_h^\sharp(\omega_2^*))_{R_\mu}=(\omega_1,\ \omega_2)_R \quad \text{for any } \omega_1,\ \omega_2\in\Gamma_h(R).$$

Proof. Let $\tau_j=A_jdz+B_jd\bar z$ $(j=1, 2)$. We have

$$(f^\sharp(\tau_1),\ f^\sharp(\tau_2^*)^*)_{R_\mu}$$
$$=-i\iint_{R_\mu}(A_1\bar A_2+B_1\bar B_2)(|(\Pi\circ f^{-1})_\zeta|^2-|(\Pi\circ f^{-1})_{\bar\zeta}|^2)d\zeta d\bar\zeta$$
$$=-i\iint_R(A_1\bar A_2+B_1\bar B_2)dzd\bar z=-(\tau_1,\ \tau_2)_R.$$

This proves the first equality. The second equality follows from the first equality and the orthogonal decomposition $\Gamma=\Gamma_h\dot+\Gamma_{eo}\dot+\Gamma_{eo}^*$.

Corollary 1. $\sigma(f(C))^*=f_h^\#(\sigma(C))^*$.

Proof. By Proposition 6 and Lemma 5, for any $\omega\in\Gamma_h$

$$(f_h^\#(\omega),\ f_h^\#(\sigma(C))^*)_{R_\mu}=(\omega,\ \sigma(C)^*)_R=(f_h^\#(\omega),\ \sigma(f(C))^*)_{R_\mu}.$$

Since $f_h^\#$ is an isomorphism between $\Gamma_h(R)$ and $\Gamma_h(R_\mu)$, we can obtain the conclusion.

Now we can show that the quasiconformal mapping f induces an $(a_i,\ b_i)$-behavior space $\Gamma_{x,\mu}(R_\mu)$ on R_μ from the behavior space $\Gamma_x(R)$ on R. We set

$$\Gamma_{x,\mu}(R_\mu)=\{f_h^\#(\omega);\ \omega\in\Gamma_x(R)\}.$$

Proposition 7. The space $\Gamma_{x,\mu}(R_\mu)$ is an $(a_i,\ b_i)$-behavior space on R_μ, i.e.
(i) $\Gamma_{x,\mu}(R_\mu)\subset\Gamma_{hse}(R_\mu)$, *(ii)* $\Gamma_{x,\mu}(R_\mu)\dot{+}\Gamma_{x,\mu}(R_\mu)^*=\Gamma_h(R_\mu)$, *(iii)* $\Gamma_{x,\mu}(R_\mu)=\overline{\Gamma_{x,\mu}(R_\mu)}$, *(iv)* $a_i\int_{A_i}\omega=b_i\int_{B_i}\omega$ *for any* $\omega\in\Gamma_{x,\mu}(R_\mu)$.

Proof. From Lemma 5, (i) and (iv) are evident and by the definition (iii) is clear. As for (ii) we first show that $\Gamma_{x,\mu}(R_\mu)$ is orthogonal to $\Gamma_{x,\mu}(R_\mu)^*$. By Proposition 6 we know that for $\omega_1,\ \omega_2\in\Gamma_x(R)$

$$(f_h^\#(\omega_1)^*,\ f_h^\#(\omega_2))_{R_\mu}=(\omega_1,\ -\omega_2^*)_R=0.$$

Next if $\omega'\in\Gamma_h(R_\mu)$ is orthogonal to $\Gamma_{x,\mu}(R_\mu)\dot{+}\Gamma_{x,\mu}(R_\mu)^*$, then for $\omega\in\Gamma_x(R)$

$$\begin{aligned}0&=(\omega',\ f_h^\#(\omega)^*)_{R_\mu}=((f^{-1})_h^\#(\omega')^*,\ (f^{-1})_h^\#\circ(-f_h^\#(\omega)))_R\\&=((f^{-1})_h^\#(\omega')^*,\ -\omega)_R.\end{aligned}$$

Hence $(f^{-1})_h^\#(\omega')^*\in\Gamma_x(R)^*$ and $(f^{-1})_h^\#(\omega')\in\Gamma_x(R)$. Thus we have $\omega'=f_h^\#\circ(f^{-1})_h^\#(\omega')\in\Gamma_{x,\mu}(R_\mu)$ must be 0. Therefore, this shows (ii).

We have meromorphic differentials $\psi_{j,x,\mu}$, $\psi_{f(p),n,x,\mu}$, $\psi_{f(p),f(q),x,\mu}$, with $\Gamma_{x,\mu}$-behavior on R_μ as in Proposition 3 and also $\Psi_j^{x,\mu}$, $\Psi_{f(p),n}^{x,\mu}$, $\Psi_{f(p),f(q)}^{x,\mu}$. We make use of the Hadamard variational method and follow Rouch[13] and Ahlfors[1]. Then we have

Lemma 6.

$$\iint_R|(\Psi_j^{x,\mu})_\zeta\circ f\zeta_z-(\Psi_j^x)_z|^2dzd\bar{z}=\iint_R|(\Psi_j^{x,\mu})_\zeta\circ f\mu\zeta_z|^2dzd\bar{z}.$$

Proof, Since $\psi_{j,x,\mu}$ has $\Gamma_{x,\mu}$-behavior, $(f^{-1})^\#\psi_{j,x,\mu}$ is equal to an element in $\Gamma_x+\Gamma_{eo}$ in a neighbourhood of the ideal boundary. By Lemma 5 $(f^{-1})^\#$ preserve periods. So by Lemma 3 $(f^{-1})^\#\psi_{j,x,\mu}-\psi_{j,x}\in\Gamma_x+\Gamma_{eo}$.
Thus we have

$$\begin{aligned}0&=((f^{-1})^\#\psi_{j,x,\mu}-\psi_{j,x},\ ((f^{-1})^\#\psi_{j,x,\mu}-\psi_{j,x})^*)_R\\&=\iint_R[(\Psi_j^{x,\mu})_\zeta\zeta_z-(\Psi_j^x)_z]dz+(\Psi_j^{x,\mu})_\zeta\zeta_{\bar{z}}d\bar{z}\\&\qquad\wedge(\overline{[(\Psi_j^{x,\mu})_\zeta\zeta_z-(\Psi_j^x)_z]dz+(\Psi_j^{x,\mu})_\zeta\zeta_{\bar{z}}d\bar{z}})^{**}\\&=-\iint_R|(\Psi_j^{x,\mu})_\zeta\zeta_z-(\Psi_j^x)_z|^2-|(\Psi_j^{x,\mu})_\zeta\zeta_{\bar{z}}|^2dzd\bar{z}.\end{aligned}$$

Since $\zeta_{\bar{z}}=\mu\zeta_z$, the assertion follows.

This guides

Lemma 7. *If* $\|\mu\|_\infty \leq k < 1$, *then*

$$\|(\Psi_j^{x,\mu})_\zeta \circ f \zeta_z \mathrm{d}z\|_R \leq \frac{1}{1-k} \|\psi_{j,x}\|_R,$$

$$\|(f^{-1})^\# \psi_{j,x,\mu} - \psi_{j,x}\|_R \leq \frac{\sqrt{2}k}{1-k} \|\psi_{j,x}\|_R.$$

Proof. By Lemma 6 we have

$$\|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z - \psi_{j,x}\|_R = \|(\Psi_j^{x,\mu})_\zeta \mu \zeta_z \mathrm{d}\bar{z}\|_R$$

$$\leq k \|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z\|_R,$$

$$\|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z\|_R - \|\psi_{j,x}\|_R \leq k \|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z\|_R.$$

Hence $\|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z\|_R \leq \frac{1}{1-k} \|\psi_{j,x}\|_R$, and

$$\|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z - \psi_{j,x}\|_R = \|(\Psi_j^{x,\mu})_\zeta \zeta_{\bar{z}} \mathrm{d}\bar{z}\|_R$$

$$\leq k \|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z\|_R$$

$$\leq \frac{1}{1-k} \|\psi_{j,x}\|_R.$$

Next we have

$$\|(f^{-1})^\# \psi_{j,x,\mu} - \psi_{j,x}\|_R^2$$

$$= i \iint_R |(\Psi_j^{x,\mu})_\zeta \zeta_z - (\Psi_j{}^x)_z|^2 + |(\Psi_j^{x,\mu})_\zeta \zeta_{\bar{z}}|^2 \mathrm{d}z \mathrm{d}\bar{z}$$

$$= \|(\Psi_j^{x,\mu})_\zeta \zeta_z \mathrm{d}z - \psi_{j,x}\|_R^2 + \|(\Psi_j^{x,\mu})_\zeta \mu \zeta_z \mathrm{d}\bar{z}\|_R^2$$

$$= 2\|(\Psi_j^{x,\mu})_\zeta \mu \zeta_z \mathrm{d}\bar{z}\|_R^2.$$

Thus

$$\|(f^{-1})^\# \psi_{j,x,\mu} - \psi_{j,x}\|_R^2 \leq \frac{2k^2}{(1-k)^2} \|\psi_{j,x}\|_R^2.$$

This completes the proof.

Similarly we obtain

Lemma 8. *Let the support of* μ *not meet a regular region* V $(p, q \in V)$. *Assume that the singularity of* $\psi_{f(p),n,x,\mu}$ *is the same as* $\psi_{p,n,x}$, *i.e., it is normalized so that* $(f^{-1})^\# \psi_{f(p),n,x,\mu} - \psi_{p,n,x} \in \Gamma(R)$. *Then we have*

$$\iint_R |(\Psi_{f(p),n}^{x,\mu})_\zeta \circ f \zeta_z - (\Psi_{p,n}^x)_z|^2 \mathrm{d}z \mathrm{d}\bar{z} = \iint_R |(\Psi_{f(p),n}^{x,\mu})_\zeta \circ f \mu \zeta_z|^2 \mathrm{d}z \mathrm{d}\bar{z},$$

$$\iint_R |(\Psi_{f(p),f(q)}^{x,\mu})_\zeta \circ f \zeta_z - (\Psi_{p,q}^x)_z|^2 \mathrm{d}z \mathrm{d}\bar{z} = \iint_R |(\Psi_{f(p),f(q)}^{x,\mu})_\zeta \circ f \mu \zeta_z|^2 \mathrm{d}z \mathrm{d}\bar{z}.$$

Remark. We can say the singularity of $\psi_{f(p),n,x,\mu}$ is also $-(\mathrm{d}\frac{1}{z^n})$.

Lemma 9. *Under the same conditions as in Lemma 8, if* $\|\mu\|_\infty \leq k < 1$, *then*

$$||(\Psi^{x,\mu}_{f(p),n})_{\zeta}\circ f\zeta_z dz||_{R-V}\leq\frac{1}{1-k}||\psi_{p,q,x}||_{R-V},$$

$$||(\Psi^{x,\mu}_{f(p),f(q)})_{\zeta}\circ f\zeta_z dz||_{R-V}\leq\frac{1}{1-k}||\psi_{p,n,x}||_{R-V},$$

$$||(f^{-1})^{\sharp}\psi_{f(p),n,x,\mu}-\psi_{p,n,x}||_{R}\leq\frac{\sqrt{2}k}{1-k}||\psi_{p,n,x}||_{R-V},$$

$$||(f^{-1})^{\sharp}\psi_{f(p),f(q),x,\mu}-\psi_{p,q,x}||_{R}\leq\frac{\sqrt{2}k}{1-k}||\psi_{p,q,x}||_{R-V}.$$

4. Variational formulas

Let $\mu(z, t)\frac{d\bar{z}}{dz}$ be a Beltrami differential with a parameter t on R and t vary a neighbourhood of zero in the complex plane. Assume that $\mu(z, t)$ ($\mu(z, 0)\equiv 0$, $||\mu(z, t)||_{\infty}<1$) is analytic with respect to t for fixed z and $\frac{\partial}{\partial t}\mu(z, t)$ is bounded and measurable. Then $R_{\mu(z,t)}=R_t$ is defined from $\mu(z, t)\frac{d\bar{z}}{dz}$ as in section 3 and also $f_{\mu(z,t)}=f_t$ is a quasiconformal homeomorphism from $R=R_0$ to R_t whose Beltrami differential is $\mu(z, t)\frac{d\bar{z}}{dz}$.

We denote meromorphic differentials with $\Gamma_{x,\mu}$-behavior on R_t by $\psi_{j,t}(\Psi_j{}^t)$, $\psi_{p,n,t}$ $(\Psi^t_{p,n})$, $\psi_{p,q,t}(\Psi^t_{p,q})$ (cf. Proposition 3), where singularities are taken at $f_t(p)$ or $f_t(q)$ and $\psi_{p,n,t}$ is assumed to have the same singularity as $\psi_{p,n,o}$, i.e. $(f_t{}^{-1})^{\sharp}\psi_{p,n,t}-\psi_{p,n,o}\in\Gamma(R_0)$. For brevity's sake we shall omit t for $t=0$ in the notations. We set

$$T_{ij}(t)=((f_t{}^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ \sigma(B_i)^*)_R+\int_{B_i}\psi_j,$$

$$S_{i,p,n}(t)=((f_t{}^{-1})^{\sharp}\psi_{p,n,t}-\psi_{p,n},\ \sigma(B_i)^*)_R+\int_{B_i}\psi_{p,n},$$

$$R_{i,p,q}(t)=((f_t{}^{-1})^{\sharp}\psi_{p,q,t}-\psi_{p,q},\ \sigma(B_i)^*)_R+\int_{B_i}\psi_{p,q}.$$

By Lemma 6 we remark

$$((f_t{}^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ \sigma(B_i)^*)_R=(\psi_{j,t}-f_t^{\sharp}\psi_j,\ \sigma(f_t(B_i))^*)_{R_t}$$
$$=(\psi_{j,t},\ \sigma(f_t(B_i))^*)_{R_t}-((f_t{}^{-1})^{\sharp}\circ f_t^{\sharp}\psi_j,\ \sigma(B_i)^*)_R$$
$$=(\psi_{j,t},\ \sigma(f_t(B_i))^*)_{R_t}-\int_{B_i}\psi_j.$$

If we allow the notation $\int_{f_t(B_i)}\psi_{j,t}$, we can write

$$T_{ij}(t)=\int_{f_t(B_i)}\psi_{j,t}=(\psi_{j,t},\ \sigma(f_t(B_i)^*))_{R_t}.$$

Similarly we can write

$$S_{i,p,n}(t)=\int_{f_t(B_i)}\psi_{p,n,t}$$

$$R_{i,p,q}(t)=\int_{f_t(B_i)}\psi_{p,q,t}.$$

Now $\sigma(B_i)$ is equal to an element in Γ_{e0} in a neighbourhood of the ideal boundary and

$$a_j\int_{A_j}(\sigma(B_i)-\bar{\psi}_i)=-a_j\delta_{ij}-b_j\int_{B_j}\bar{\psi}_i+a_j\delta_{ij}$$
$$=b_j\int_{B_j}(\sigma(B_i)-\bar{\psi}_i).$$

By Lemma 3 $\sigma(B_i)-\bar{\psi}_i\in\Gamma_x$. Also by Lemma 5

$$a_j\int_{A_j}((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j)=b_j\int_{B_j}((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j),$$

and $(f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j$ is equal to an element in $\Gamma_x+\Gamma_{e0}$ in a neighbourhood of the ideal boundary. Hence by Lemma 3 we know $(f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j\in\Gamma_x+\Gamma_{e0}$ and

$$T_{ij}(t)=((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ \sigma(B_i)^*)_R+\int_{B_i}\psi_j$$
$$=((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ (\sigma(B_i)-\bar{\psi}_i)^*)_R$$
$$+((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ \bar{\psi}_i^*)_R+\int_{B_i}\psi_j$$
$$=((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ \bar{\psi}_i^*)_R+\int_{B_i}\psi_j.$$

We can write

$$T_{ij}(t)=((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ \bar{\psi}_i^*)_R+\int_{B_i}\psi_j,$$
$$S_{i,p,n}(t)=((f_t^{-1})^{\sharp}\psi_{p,n,t}-\psi_{p,n},\ \bar{\psi}_i^*)_R+\int_{B_i}\psi_{p,n},$$
$$R_{i,p,q}(t)=((f_t^{-1})^{\sharp}\psi_{p,q,t}-\psi_{p,q},\ \bar{\psi}_i^*)_R+\int_{B_i}\psi_{p,q}.$$

We have the following variational formula.

Lemma 10. Let $\|\mu(z,t)\|_\infty=\operatorname{esssup}_z|\mu(z,t)|=k(t)\le k<1$. *If* $\overline{\lim_{t\to 0}}\dfrac{k(t)}{|t|}<\infty$, *then*

$$\frac{d}{dt}T_{ij}(t)|_{t=0}=\iint_R(\Psi_i)_z(\Psi_j)_z\frac{\partial}{\partial t}\mu(z,t)|_{t=0}dzd\bar{z}.$$

Proof. We have

$$T_{ij}(t)-T_{ij}(0)=((f_t^{-1})^{\sharp}\psi_{j,t}-\psi_j,\ \bar{\psi}_i^*)_R$$
$$=\iint_R((\Psi_j^t)_\zeta\zeta_z-(\Psi_j)_z)dz+(\Psi_j^t)_\zeta\zeta_{\bar{z}}d\bar{z}\wedge-(\Psi_i)_z dz$$
$$=\iint_R(\Psi_i)_z(\Psi_j^t)_\zeta\mu\zeta_z dzd\bar{z}$$
$$=\iint_R(\Psi_i)_z(\Psi_j)_z\mu dzd\bar{z}+\iint_R(\Psi_i)_z((\Psi_j^t)_\zeta\zeta_z-(\Psi_j)_z)\mu dzd\bar{z}.$$

By the way, from Lemma 7,

$$\iint_R(\Psi_i)_z((\Psi_j^t)_\zeta\zeta_z-(\Psi_j)_z)\mu dzd\bar{z}$$

$$\leq k(t)\|\psi_i\|_R\|(\Psi_j^t)_\zeta\zeta_z dz-\psi_j\|_R$$
$$\leq\frac{k(t)^2}{1-k(t)}\|\psi_i\|_R\|\psi_j\|_R.$$

Since $\lim_{t\to 0} k(t)^2/|t|(1-k(t))=0$,

$$\frac{d}{dt}T_{ij}(t)|_{t=0}=\lim_{t\to 0}\frac{T_{ij}(t)-T_{ij}(0)}{t}$$
$$=\lim_{t\to 0}\iint_R(\Psi_i)_z(\Psi_j)_z\frac{\mu(z,t)}{t}dzd\bar{z}$$
$$=\iint_R(\Psi_i)_z(\Psi_j)_z\frac{\partial}{\partial t}\mu(z,t)|_{t=0}dzd\bar{z}.$$

Q.E.D..

Proposition 8. Assume that $\lim_{\tau\to 0}\|\mu(z,t+\tau)-\mu(z,t)\|_\infty/|\tau|<\infty$. *Then*

$$\frac{d}{dt}T_{ij}(t)=\iint_R(\Psi_i^t)_\zeta\circ f_t(\Psi_j^t)_\zeta\circ f_t\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2dzd\bar{z}.$$

Proof. Note that

$$T_{ij}(t+\tau)-T_{ij}(t)=((f_{t+\tau}^{-1})^\sharp\psi_{j,t+\tau}-(f_t^{-1})^\sharp\psi_{j,t},\ \sigma(B_i)^*)_R$$
$$=(f_t^\sharp\circ(f_{t+\tau}^{-1})^\sharp\psi_{j,t+\tau}-\psi_{j,t},\ \sigma(f_t(B_i))^*)_{R_t}.$$

The Beltrami differential of $f_{t+\tau}\circ f_t^{-1}$ is

$$\frac{\zeta_z}{(\bar{\zeta})_{\bar{z}}}\frac{\mu(z,t+\tau)-\mu(z,t)}{1-\overline{\mu(z,t)}\mu(z,t+\tau)}\circ f_t^{-1}\frac{d\bar{\zeta}}{d\zeta}.$$

If we denote this by $\nu(\zeta,\tau)\dfrac{d\bar{\zeta}}{d\zeta}$, by Lemma 10

$$\frac{d}{dt}T_{ij}(t)=\iint_{R_t}(\Psi_i^t)_\zeta(\Psi_j^t)_\zeta\lim_{\tau\to 0}\frac{\nu(\zeta,\tau)}{|\tau|}d\zeta d\bar{\zeta}$$
$$=\iint_{R_t}(\Psi_i^t)_\zeta(\Psi_j^t)_\zeta\frac{\zeta_z}{(\bar{\zeta})_{\bar{z}}}\frac{\mu_t(z,t)}{1-|\mu(z,t)|^2}\circ f_t^{-1}d\zeta d\bar{\zeta}$$
$$=\iint_R(\Psi_i^t)_\zeta\circ f_t(\Psi_j^t)_\zeta\circ f_t\frac{\mu_t(z,t)}{1-|\mu(z,t)|^2}\frac{\zeta_z}{(\bar{\zeta})_{\bar{z}}}\frac{\partial(\zeta,\bar{\zeta})}{\partial(z,\bar{z})}dzd\bar{z}$$
$$=\iint_R(\Psi_i^t)_\zeta\circ f_t(\Psi_j^t)_\zeta\circ f_t\mu_t(z,t)\zeta_z{}^2dzd\bar{z}.$$

Similarly we have

Proposition 9. Assume that the support of $\mu(z,t)$ *does not meet a regular region* V *which contains* p *and* q, *and that* $\overline{\lim_{\tau\to 0}}\|\mu(z,t+\tau)-\mu(z,t)\|_\infty/|\tau|<\infty$. *Then*

$$\frac{d}{dt}S_{i,p,n}(t)=\iint_R(\Psi_{p,n}^t)_\zeta\circ f_t(\Psi_i^t)_\zeta\circ f_t\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2dzd\bar{z},$$
$$\frac{d}{dt}R_{i,p,q}(t)=\iint_R(\Psi_{p,q}^t)_\zeta\circ f_t(\Psi_i^t)_\zeta\circ f_t\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2dzd\bar{z}$$

Under these conditions, the variational formulas show the following.

Proposition 10. The functions $T_{ij}(t)$, $S_{i,p,n}(t)$ *and* $R_{i,p,q}(t)$ *are analytic functions for variable* t.

From Proposition 5, we have

Proposition 11. The functions $\Psi_j{}^t\circ f_t(p, q)$ *and* $\frac{d^m}{dz^m}\Psi_j{}^t\circ f_t(p, q)$ *are analytic functions for variable* t *and they are analytic for variables* $p, \neq q\in V$ *and* t, *where* V *does not meet the support of* $\mu(z, t)$.

Let V be a regular region which contains p, q and does not meet $\cup(A_i\cup B_i)$. Further, assume that the support of $\mu(z, t)$ does not meet p', q' and $\bar{V}$. Since f_t is conformal at p' and q', $(f_t^{-1})^\sharp\psi_{p',q',t}$ has the same singularity as $\psi_{p',q'}$. Hence $(f_t^{-1})^\sharp\psi_{p',q',t}-\psi_{p',q'}$ belongs to Γ and, by Lemma 3, to $\Gamma_x+\Gamma_{eo}$. We can consider the inner product

$$((f_t^{-1})^\sharp\psi_{p',q',t}-\psi_{p',q'}, \overline{\psi_{p,q}{}^*})_{R-V}.$$

The differential d log $(z'-\Pi(p))/(z'-\Pi(q))$ in a parametric disk $\bar{V}'\subset V$ ($p, q\in V'$ and z' is the local parameter) is extended to a C^1-closed differential $\sigma_{p,q}$ whose support is contained in V. Then $\psi_{p,q}-\sigma_{p,q}$ belongs to $\Gamma_x+\Gamma_{eo}$. We have

$$\begin{aligned}
&((f_t^{-1})^\sharp\psi_{p',q',t}-\psi_{p',q'}, \overline{\psi_{p,q}{}^*})_{R-V}\\
&=((f_t^{-1})^\sharp\psi_{p',q',t}-\psi_{p',q'}, \overline{(\psi_{p,q}-\sigma_{p,q})^*})_{R-V}\\
&=-((f_t^{-1})^\sharp\psi_{p',q',t}-\psi_{p',q'}, \overline{(\psi_{p,q}-\sigma_{p,q})^*})_V\\
&=\int_{\partial V}(\Psi^t_{p',q',t}\circ f_t-\Psi_{p',q'})(\psi_{p,q}-\sigma_{p,q})\\
&=2\pi i\{\Psi^t_{p',q'}\circ f_t(p, q)-\Psi_{p',q'}(p, q)\}.
\end{aligned}$$

Thus we can write $\Psi^t_{p',q'}\circ f_t(p, q)$ as

$$\Psi^t_{p',q'}\circ f_t(p, q)=\frac{1}{2\pi i}((f_t^{-1})^\sharp\psi_{p',q',t}-\psi_{p',q'}, \overline{\psi_{p,q}{}^*})_{R-V}+\Psi_{p',q'}(p, q).$$

Further we have

$$\frac{d^m}{dz^m}\Psi^t_{p',q'}\circ f_t(p, q)=\frac{(m-1)!}{2\pi i}((f_t^{-1})^\sharp\psi_{p',q',t}-\psi_{p',q'}, \overline{\psi_{p,m}{}^*})_{R-V}+\frac{d^m}{dz^m}\psi_{p',q'}(p, q).$$

If we also assume that $\psi_{p',n,t}$ has the same singularity as $\psi_{p',n}$, then we can write as

$$\Psi^t_{p',n}\circ f_t(p, q)=\frac{1}{2\pi i}((f_t^{-1})\psi_{p',n,t}-\psi_{p',n}, \overline{\psi_{p,q}{}^*})_{R-V}+\Psi_{q',n}(p, q),$$

$$\frac{d^m}{dz^m}\Psi^t_{p',n}\circ f_t(p, q)=\frac{(m-1)!}{2\pi i}((f_t^{-1})^\sharp\psi_{p',n,t}-\psi_{p',n}, \overline{\psi_{p,m}{}^*})_{R-V}+\frac{d^m}{dz^m}\Psi_{p',n}(p, q).$$

Under these circumstances, we can obtain the following.

Proposition **12.** *If* $\overline{\lim_{\tau\to 0}}\, ||\mu(z, t+\tau)-\mu(z, t)||_\infty/|\tau|<\infty$, *then*

$$\frac{d}{dt}\Psi^t_{p',q'}\circ f_t(p, q)=\frac{1}{2\pi i}\iint_R (\Psi^t_{p',q'})_\zeta\circ f_t(\Psi^t_{p,q})_\zeta\circ f_t\frac{\partial}{\partial t}\mu(z, t)\zeta_z{}^2 dz d\bar{z},$$

$$\frac{d}{dt}\left(\frac{d^m}{dz^m}\Psi^t_{p',q'}\circ f_t(z, q)|_{z=0}\right)$$

$$=\frac{(m-1)!}{2\pi i}\iint_R (\Psi^t_{p',q'})_\zeta\circ f_t(\Psi^t_{p,m})_\zeta\circ f_t\frac{\partial}{\partial t}\mu(z, t)\zeta_z{}^2 dz d\bar{z},$$

$$\frac{d}{dt}\Psi^t_{p',n}\circ f_t(p, q)=\frac{1}{2\pi i}\iint_R (\Psi^t_{p',n})_\zeta\circ f_t(\Psi^t_{p,q})_\zeta\circ f_t\frac{\partial}{\partial t}\mu(z, t)\zeta_z{}^2 dz d\bar{z},$$

$$\frac{d}{dt}\left(\frac{d^m}{dz^m}\Psi^t_{p',n}\circ f_t(z, q)|_{z=0}\right)$$

$$=\frac{(m-1)!}{2\pi i}\iint_R (\Psi^t_{p',n})_\zeta\circ f_t(\Psi^t_{p,m})_\zeta\circ f_t\frac{\partial}{\partial t}\mu(z, t)\zeta_z{}^2 dz d\bar{z}.$$

The functions $\Psi^t_{p',q'}\circ f_t(p, q)$, $\frac{d^m}{dz^m}\Psi^t_{p',q'}\circ f_t(p, q)$, $\Psi^t_{p',n}\circ f_t(p, q)$ *and* $\frac{d^m}{dz^m}\Psi^t_{p',n}\circ f_t(p, q)$ *are analytic for variable* t. *Hence* $\Psi^t_{p',q'}\circ f_t(p, q)$ *and* $\Psi^t_{p',n}\circ f_t(p, q)$ *are analytic for variables* $p, \neq q\in V$ *and* t.

5. Examples

Example **1.** Let R_t be a two-sheeted covering surface of a fixed region in the complex plane with a parameter t varied in the unit disk and let $\{a_n(t)\}$ denote a countable number of branch points of R_t. We take disks $V_n=\{z;\ |z-a_n(0)|<r_n\}$ which are disjoint. Assume that $a_n(t)$ are holomorphic and $|a_n(t)-a_n(0)|\leq kr_n(k<1)$. We consider a function f_t on C;

$$f_t(z)=\begin{cases}\dfrac{r_n{}^2(z+a_n(t)-2a_n(0))}{r_n{}^2+(a_n(t)-a_n(0))(\overline{z-a_n(0)})}+a_n(0) & \text{on every } \bar{V}_n\\ z \text{ outside of } \bigcup_n V_n.\end{cases}$$

Then $f_t(a_n(0))=a_n(t)$, $f_t(z)=z$ on every $\{z;\ |z-a_n(0)|=r_n\}$ and

$$\frac{(f_t)_{\bar z}}{(f_t)_z}=\begin{cases}\dfrac{-(a_n(t)-a_n(0))(z+a_n(t)-2a_n(0))}{r_n{}^2+(a_n(t)-a_n(0))(\overline{z-a_n)(0)})} & \text{on every } V_n\\ 0 \text{ outside of } \bigcup_n V_n,\end{cases}$$

which is analytic for variable t. We also have $|(f_t)_{\bar z}/(f_t)_z|<k$. We can regard f_t as a quasiconformal homeomorphism from R to R_t whose Beltrami coefficient $\mu(z, t)$ is analytic for variable t $\left(\frac{\partial}{\partial t}\mu(z, t)\text{ is bounded and measurable}\right)$. The $\mu(z, t)$ satisfies $\overline{\lim_{\tau\to 0}}\,|\mu(z, t+\tau)-\mu(z, t)|/|\tau|<\infty$. Thus by Proposition 10 $T_{ij}(t)$, $S_{i,p,n}(t)$ and $R_{i,p,q}(t)$ $(p, q\notin \bigcup_n V_n)$ are analytic functions for variable t.

Example **2.** Let R be a finite bordered Riemann surface (with a finite genus) whose

boundary ∂R consists of a finite number of compact analytic curves C_j. We denote by $V_j=\{z_j;\ \rho_j<|z_j|<1\}$ a ring domain whose boundary $\{z_j;\ |z_j|=1\}$ is C_j. Let $f_j(z_j, t)$ be analytic with respect to z_j and t on a neighbourhood of $\{|z_j|=1\}\times\{0\}$, $f_j(z_j, 0)=z_j$ and injective on $\{|z_j|=1\}$ for a fixed t.
We can assume that for a sufficiently small t

$$F_j(z_j, t)=\frac{\rho_j(1-|z_j|)z_j}{(1-\rho_j)|z_j|}+\frac{|z_j|-\rho_j}{(1-\rho_j)}f_j\left(\frac{z_j}{|z_j|}, t\right)$$

is a quasiconformal homeomorphism from V_j to a ring domain $V_j(t)$. We regard $(R-\bigcup_n V_n)\cup\bigcup_n V_n(t)$ as a Riemann surface R_t and

$$F_t=\begin{cases}\text{identity mapping on } R-\bigcup_n V_n\\ F_j(z_j, t)\text{ on every } V_j\end{cases}$$

a quasiconformal homeomorphism from R to R_t. The Beltrami coefficient $\mu(z, t)$ of F is analytic with respect to t and satisfies the condition of Propositions 8 and 9. Hence $T_{ij}(t)$, $S_{i,p,n}(t)$ and $R_{i,p,q}(t)$ are analytic functions for variable t.

These give examples that each element of a period matrix is holomorphic if branch points or boundary curves vary holomorphically.

Studies of General Education,
College of Technology,
Kyoto Technical University,
Matsugasaki, Sakyo-ku, Kyoto **606.**

References

1) L. V. Ahlfors, "Analytic functions", 45–66, Princeton Univ. Press (1960).
2) L. V. Ahlfors, "Lecture on quasiconformal mappings", Van Nostrand (1966).
3) L. V. Ahlfors & L. Sario, "Riemann surfaces", Princeton Univ. Press (1960).
4) L. Bers, *Bull. Amer. Math. Soc.*, **67**, 206–210 (1961).
5) C. Constantinescu & A. Cornea, "Ideale Ränder Riemannscher Flächen", Springer-Verlag (1963).
6) Y. Kusunoki, *Hokkaido Math. J.*, **10**, 435–446 (1981).
7) Y. Kusunoki & F. Maitani, Math. Zeit., 181, 435–450 (1982).
8) Y. Kusunoki & M. Taniguchi, *Ann. Acad. Sci. Fenn. Ser. A.I. Math.*, **5**, 207–226 (1980).
9) F. Maitani, *J. Math. Kyoto Univ.*, **20**, 661–689 (1980).
10) A. Marden, *Ann. Acad. Sci. Fenn.*, **359**, (1965).
11) C. D. Minda, *Trans. Amer. Math. Soc.*, **195**, 365–381 (1974).
12) C. Patt, *J. d'Analyse Math.*, **11**, 221–247 (1963).
13) H. E. Rauch, *Comm. Pure and Appl. Math.*, **12**, 543–560 (1959).
14) M. Shiba, *J. Math. Kyoto Univ.*, **11**, 495–525 (1971).