Directional Stability Radius: A Stability Analysis Tool for Uncertain Polynomial Systems

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Abstract—Coefficients of characteristic polynomials for stable parametrically uncertain systems are allowed to perturb to some extent for stability. Stability radius is a useful tool to assess the allowance of the stability for the systems. To enhance its usefulness, we modify stability radius so that it takes into account of given restricted perturbations, which we call directional stability radius. For an application, we show shifted-Hurwitz stability conditions and a stability analysis method for interval polynomial systems using the directional stability radii.

Index Terms—Delta-operator systems, directional stability radius, interval polynomial, shifted-Hurwitz stability, stability analysis tool.

I. Introduction

Parameters of systems may have uncertainties owing to errors in modeling. Stability analysis for the systems considering the real parametric uncertainties has been carried out, and various results and techniques have been thus far developed [4], [5]. The most notable among them is extreme point results, which enable to dramatically save computational cost and time in testing stability of polynomials with polytopic uncertainties. Still, save for strong extreme point results such as celebrated Kharitonov's theorem [2], the computation becomes far-from-cheap when the degree of polynomials or the number of vertex polynomials increases. The aim of this note is a proposal of a new analysis tool that can overcome this difficulty. The key idea is to consider a specific stability radius, which is the conventional stability radius being under restrictions defined by the uncertainty range. Since this directional stability radius ensures stability in bulk in the coefficient space, it can reduce the cost in treating systems where the strong extreme point results do not hold. As an application of this concept, we focus on systems and their stability for which a weak extreme point result is true: shifted-Hurwitz property of interval systems. Stability analysis method for the systems is checking stability for all vertex polynomials [1]. Other approaches applicable to this case do exist [4]. One is using the ordinary stability radius with the weighted l_∞ norm, where the weights are determined according to width of the intervals. The other is checking stability at one vertex followed by a check of maximal phase difference across the vertices over the stability boundary. In this note, we provide a useful alternative. By applying the proposed stability radius to the setting, the stability of the systems can be effectively checked. The organization of this note is as follows. In Section II, directional stability radius is defined as a stability analysis tool for uncertain systems. As an application, stability conditions are shown for interval systems in terms of directional stability radius and an analysis method for the systems is proposed in Section III. In Section IV, numerical examples are demonstrated to show the efficiency of our method. Finally, concluding remarks are given in Section V.

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II. Directional Stability Radius

This note deals with stability analysis for systems of which characteristic polynomials

$$f(s, a) = a_0 + a_1 s + \cdots + a_n s^n \quad (1)$$

have uncertainties in their coefficients $a = [a_0, a_1, \ldots, a_n]^T$. From the viewpoint of stability for uncertain systems, it is important to know how large the coefficient perturbations of a stable polynomial could be so that they do not destroy stability. Stability radius for a polynomial can be an estimator of stability margin against the coefficient perturbations. Consider an $(n+1)$-dimensional space in which a point a corresponds to a set of the coefficients of the polynomial (1). This space is called coefficient space. In the space, the stability region is represented as a set

$$S = \{a: f(s, a) \text{ is stable}\} \quad (2)$$

and the boundary of S is designated as B. The stability radius is the minimum size of the perturbations from a which destroy its stability. Namely, the stability radius [4] $\rho(a)$ is defined by

$$\rho(a) := \inf_{b \in B} \|b - a\| \quad (3)$$

where $\|\cdot\|$ means Euclidean norm. The hyperball centered at a with radius $\rho(a)$

$$R(a) = \{a': \|a' - a\| < \rho(a)\} \quad (4)$$

is in the stability region. However $\rho(a)$ is conservative when perturbation directions in the space are restricted, because $R(a)$ guarantees the stability for any directions from a. When perturbation directions are represented as the region surrounded by m planes

$$\{a': D(a' - a) \leq 0\}, \qquad D \in R^{m \times (n+1)} \quad (5)$$

taking account of this restriction, we modify the stability radius as the following form:

$$\rho^*(a) := \inf_{b \in B} \|b - a\| \quad (6)$$
$$\text{subject to } D(b - a) \leq 0. \quad (7)$$

$\rho^*(a)$ is called directional stability radius, and then the restricted hyperball

$$R^*(a) = \{a': \|a' - a\| < \rho^*(a), D(a' - a) \leq 0\} \quad (8)$$

is in the stability region. Therefore, by choosing D properly, $\rho^*(a)$ can measure stability margin for any specified directions. Note that $\rho^*(a) \geq \rho(a)$. Using this measure for stability check of uncertain polynomials is the central conceptual idea of this paper and it proves to be powerful in many situations. To be more specific, we will apply it to the shifted-Hurwitz stability test for interval polynomials.

III. Shifted-Hurwitz Stability Analysis for Interval Systems

A. Directional Stability Radius for Interval Polynomials

For parametrically uncertain systems, the coefficients of characteristic polynomials are designed to be allowed to perturb in a region in the coefficients space. Interval polynomials are ones that the perturbation region of the coefficients of polynomials (1) is defined by

$$\alpha_i \leq a_i \leq \beta_i, \qquad i = 0, \ldots, n. \quad (9)$$

We consider damping performance of the systems besides their stability. For such case, the stability region on the complex plane is described as

$$\{x + jy: x, y \in \Re, x < -h\} \quad (10)$$

where $h > 0$ expresses damping performance. It is called shifted-Hurwitz stability. The stability of interval polynomials (1), (9) means that all zeros of the polynomial are in the stability region (10) on the complex plane for any values of coefficients satisfying (9). In the coefficient space, a set of the coefficients of an interval polynomial is represented as

$$Q = \{a: \alpha_i \leq a_i \leq \beta_i, \quad i = 0, \ldots, n\} \quad (11)$$

giving a hyperbox. The stability of the interval polynomial means that Q is included in S in the space. It is known that a stability property, called weak extreme point result is satisfied for shifted-Hurwitz stability of interval polynomials [1]. Here, by "weak," we mean that stability conditions depend on n, while by "strong" they do not.

Lemma 1: The interval polynomial (1), (9) is shifted-Hurwitz stable if and only if all the vertex polynomials, that is, the polynomial whose coefficients take boundary values of the interval (9) such as

$$a_i = \alpha_i \text{ or } \beta_i, \qquad i = 0, \ldots, n \quad (12)$$

are sifted-Hurwitz stable.

In the coefficient space, each vertex polynomial corresponds to a vertex of Q and the number of the vertex polynomials is 2^{n+1}. We label the coefficients of vertex polynomials as a_{V_j}, $j = 1, \ldots, 2^{n+1}$. Checking the stability of each vertex polynomial $f(s, a_{V_j})$ for large n is a tough work as the number of the vertex polynomials suggests. Here comes the directional stability radius $\rho^*(a_{V_j})$. Stability margin at a vertex a_{V_j} can be used to check the stability of the other vertices, and thus reduce the work for stability analysis. The perturbation directions D in (7) for the vertex a_{V_j} is represented, in this case, by

$$D_{V_j} = \operatorname{diag}(p_0, p_1, \ldots, p_n) \in R^{(n+1)\times(n+1)}$$
$$p_i = \begin{cases} +1, & \text{for } a_i = \beta_i \\ -1, & \text{for } a_i = \alpha_i \end{cases}, \qquad i = 0, \ldots, n \quad (13)$$

which means that $(n+1)$ planes containing the surfaces of Q compose the region of restricted perturbations. To obtain $\rho^*(a_{V_j})$, we have to compute $\rho(a_{V_j})$ and this can be carried out in the same way as "shiftless" Hurwitz case [3]. First, the boundary B in (6) for shifted-Hurwitz stability will be described. The boundary of the region (10) is separated into three parts: B_1, B_2^y, and B_3.

1) The polynomial (1) has a zero on the real axis $x = -h$

$$B_1 = \{\Phi_1 t_n: \forall t_n \in R^n\}$$
$$\Phi_1 = \begin{bmatrix} h & & 0 \\ 1 & \ddots & \\ & \ddots & h \\ 0 & & 1 \end{bmatrix} \in R^{(n+1)\times n}. \quad (14)$$

2) The polynomial (1) has two conjugate zeros $-h \pm y$

$$B_2^y = \{\Phi_2^y t_{n-1}: \forall t_{n-1} \in R^{n-1}\}$$
$$\Phi_2^y = \begin{bmatrix} h^2 + y^2 & & 0 \\ 2h & \ddots & \\ 1 & \ddots & h^2 + y^2 \\ & \ddots & 2h \\ 0 & & 1 \end{bmatrix} \in R^{(n+1)\times(n-1)}. \quad (15)$$

3) The polynomial (1) has a zero at infinity, that is a degree dropping

$$B_3 = \{\Phi_3 t_n : \forall t_n \in R^n\}$$

$$\Phi_3 = \begin{bmatrix} 1 & & 0 \\ 0 & \ddots & \\ & \ddots & 1 \\ 0 & & 0 \end{bmatrix} \in R^{(n+1)\times n}. \quad (16)$$

Then, substituting B_1, B_2^y, and B_3 given, respectively, by (14)–(16) into the constrained optimization problem (6), (7) leads to quadratic programming problems

$$\rho_1^*(a_{V_j})^2 = \inf_{t_n} \|\Phi_1 t_n - a_{V_j}\|^2 \quad (17)$$

$$\text{subject to } D_{V_j}(\Phi_1 t_n - a_{V_j}) \le 0 \quad (18)$$

$$\rho_2^{*y}(a_{V_j})^2 = \inf_{t_{n-1}} \|\Phi_2^y t_{n-1} - a_{V_j}\|^2 \quad (19)$$

$$\text{subject to } D_{V_j}(\Phi_2^y t_n - a_{V_j}) \le 0 \quad (20)$$

$$\rho_3^*(a_{V_j})^2 = \inf_{t_n} \|\Phi_3 t_n - a_{V_j}\|^2 \quad (21)$$

$$\text{subject to } D_{V_j}(\Phi_3 t_n - a_{V_j}) \le 0 \quad (22)$$

respectively. Note that, since $\rho_2^{*y}(a_{V_j})^2$ defined by (19) and (20) is parametrized by y, the following one-dimensional search problem:

$$\rho_2^*(a_{V_j}) = \min_{0<y<\infty} \rho_2^{*y}(a_{V_j}) \quad (23)$$

is solved. Finally, the directional stability radius at a_{V_j} is computed by

$$\rho^*(a_{V_j}) = \min\{\rho_1^*(a_{V_j}), \rho_2^*(a_{V_j}), \rho_3^*(a_{V_j})\}. \quad (24)$$

B. Stability Conditions and Analysis Method

Using the directional stability radius $\rho^*(a_{V_j})$, we can derive a stability condition effective for the analysis of the systems. A set of labels for the vertices at which directional stability radii are computed is defined by I_{sub}, which is some subset of $\{1, \ldots, 2^{n+1}\}$.

Theorem 1: If all vertices a_{V_j} are included by a union of some directional stability balls $\bigcup_{k\in I_{\text{sub}}} R^*(a_{V_k})$, namely, for all $j \in \{1, \ldots, 2^{n+1}\}$, there exists $k \in I_{\text{sub}}$ such that the inequalities

$$\|a_{V_j} - a_{V_k}\| < \rho^*(a_{V_k}) \quad (25)$$

hold, then the interval polynomial (1), (9) is shifted-Hurwitz stable.

Proof: It is derived from Lemma 1 and the fact that the inside of $R^*(a_{V_k})$ are the stability regions. ■

The stability condition given in the above theorem could give us possibilities of reducing computational burden for the stability test, which normally entails 2^{n+1} point-wise checks. Making use of this stability condition, we propose an algorithm based on branch-and-bound method to check the stability efficiently. For each vertex a_{V_j}, define $x_{V_j} = [x_0^{V_j}, x_1^{V_j}, \ldots, x_n^{V_j}]^T$ by

$$x_i^{V_j} = \begin{cases} 0, & \text{for } a_i^{V_j} = \alpha_i \\ 1, & \text{for } a_i^{V_j} = \beta_i \end{cases}, \qquad i = 0, \ldots, n. \quad (26)$$

Then, (25) is replaced by the inequalities

$$\sum_{i=0}^{n} (\beta_i - \alpha_i)^2 (x_i^{V_j} \oplus x_i^{V_k}) < \rho^*(a_{V_k})^2 \quad (27)$$

where $\oplus$ refers to "exclusive or." Now, the problem of finding x_{V_j} satisfying (27) is defined by

$$P_0 \quad \left| \begin{array}{ll} \text{find} & x_{V_j},\ j \in \{1, \ldots, 2^{n+1}\} \\ \text{subject to} & \sum_{i=0}^{n} (\beta_i - \alpha_i)^2 (x_i^{V_j} \oplus x_i^{V_k}) < \rho^*(a_{V_k})^2, \\ & k \in I_{\text{sub}} \end{array} \right. \quad (28)$$

which is a combinatorial problem with respect to x_{V_j} (problem P_0). As trivial cases,

(Condition 1) if for some $k \in I_{\text{sub}}$, the inequality

$$\sum_{i=0}^{n} (\beta_i - \alpha_i)^2 \le \rho^*(a_{V_k})^2 \quad (29)$$

holds, some directional stability ball includes all vertices, that is, the solution to P_0 is all x_{V_j}. Also,

(Condition 2) if for all $k \in I_{\text{sub}}$, the inequalities

$$0 \ge \rho^*(a_{V_k})^2 \quad (30)$$

hold, any directional stability ball includes no vertices, thus, the solution is none. These conditions can be applied to partial problems of P_0 in which some variables x_{V_j} are fixed to 0 or 1. Now, with the above two formulated terminating conditions, the algorithm to solve P_0 is as follows.

Algorithm 1 (Algorithm for Solving the Problem P_0):

0) A set of generated partial problems is denoted by $\mathcal{P}$, and a set of solutions of the original problem P_0 is denoted by $\mathcal{X}$.
1) Initially set $\mathcal{P} \to \{P_0\}$ and $\mathcal{X} \to \emptyset$.
2) If $\mathcal{P} = \emptyset$, go to step 7), else, go to step 3).
3) Pick a partial problem P_s from $\mathcal{P}$. $\mathcal{P} \to \mathcal{P} - P_s$. For P_s whose d variables are fixed $x_i^{V_j} = \hat{x}_i^{V_j}$ (0 or 1), $i = 0, \ldots, d-1$, if condition 1

$$\exists k \in I_{\text{sub}}: \sum_{i=d}^{n} (\beta_i - \alpha_i)^2 \le \rho^*(a_{V_k})^2 - \sum_{0}^{d-1} (\beta_i - \alpha_i)^2 (\hat{x}_i^{V_j} \oplus x_i^{V_k}) \quad (31)$$

holds, go to step 4), if condition 2

$$0 \ge \rho^*(a_{V_k})^2 - \sum_{i=0}^{d-1} (\beta_i - \alpha_i)^2 (\hat{x}_i^{V_j} \oplus x_i^{V_k}) \qquad \forall k \in I_{\text{sub}} \quad (32)$$

hold, go to step 5), and if both conditions do not hold, go to step 6).
4) Add all solutions of P_s whose number is 2^{n+1-d} to $\mathcal{X}$, $\mathcal{X} \to \mathcal{X} \cup \{x_{V_j} : x_i^{V_j} = \hat{x}_i^{V_j}, i = 0, \ldots, d-1, x_i^{V_j} = 0, 1, i = d, \ldots, n\}$. go to step 2).
5) Solutions of P_s are none. go to step 2).
6) Make two partial problems of P_s, P_{s1} and P_{s2} whose $(d+1)$th variable $x_d^{V_j}$ are fixed to 0 or 1, and add them to $\mathcal{P}$. $\mathcal{P} \to \mathcal{P} \cup \{P_{s1}, P_{s2}\}$. go to step 2).
7) Computation is finished. The obtained $\mathcal{X}$ is a set of solutions of the original problem P_0.

As a result of applying algorithm 1, if all the vertices x_{V_j} are included in $\mathcal{X}$, it can be concluded immediately that the interval polynomial is stable. The vertices not included in $\mathcal{X}$ remain open about whether it is stable or not. For these vertex polynomials, we must test their stability individually with a conventional method.

In addition to the aforementioned condition and algorithm, the directional stability radius $\rho^*(a_{V_j})$ can lead a instability condition for the systems.

Theorem 2: If some directional stability ball $R^*(a_{V_j})$ is included in Q, that is, there exists $j \in \{1, \ldots, 2^{n+1}\}$ such that

$$\rho^*(a_{V_j}) \le \min_{i=0,\ldots,n} |\beta_i - \alpha_i| \tag{33}$$

holds, the interval polynomial (1), (9) is unstable in the sense of shifted-Hurwitz stability.

Proof: We can assume that (33) holds on a_1 without loss of generality, considering symmetry of the shape of Q. The value of the right-hand side of (33) is the length of the shortest edge of the hyperbox Q. Hence, the assumption that $\rho^*(a_1)$ satisfies (33) indicates $R^*(a_1) \subset Q$. On the other hand, from the definition of directional stability radius, the surface of $R^*(a_1)$ must touch the boundary of stability region B, namely $R^*(a_1) \cap B \neq \emptyset$. The two results give a conclusion that $Q \cap B \neq \emptyset$. Therefore, the interval polynomial is unstable. ■

From Theorem 2, if some vertex a_{V_k}, $k \in I_{sub}$ satisfies (33), we can conclude instability of the system without carrying out Algorithm 1.

IV. Numerical Examples

For the purpose of illustrating our stability analysis method previously mentioned, we analyze shifted-Hurwitz stability for the third-degree interval polynomial

$$\begin{aligned} f(s, a) &= a_0 + a_1 s + a_2 s^2 + a_3 s^3 \\ 87 &\le a_0 \le 88 \quad 216 \le a_1 \le 217 \\ 180 &\le a_2 \le 181 \quad 49 \le a_3 \le 50 \end{aligned} \tag{34}$$

for some values of the shift: $h = 0.90$, 0.95, and 1.0. The vertices where directional stability radii are first computed are chosen from the following ones, whose stability gives an exact stability property for interval polynomials at $h = 0$ [2]. Namely, when $h = 0$, the weak extreme point result (Lemma 1) reduces to the strong extreme point result (Lemma 2).

Lemma 2: A necessary and sufficient condition for Hurwitz stability for interval polynomials (1), (9) is that the four Kharitonov polynomials given by

$$\begin{aligned} a_{K_1} &= [\alpha_0, \alpha_1, \beta_2, \beta_3, \ldots]^T \\ a_{K_2} &= [\alpha_0, \beta_1, \beta_2, \alpha_3, \ldots]^T \\ a_{K_3} &= [\beta_0, \beta_1, \alpha_2, \alpha_3, \ldots]^T \\ a_{K_4} &= [\beta_0, \alpha_1, \alpha_2, \beta_3, \ldots]^T \end{aligned} \tag{35}$$

are Hurwitz stable.

Due to this lemma, the stability of the four Kharitonov polynomials at $h (>0)$ could be regarded as necessary conditions which are near to a necessary and sufficient one for small h. Therefore, we choose the four vertices in (35) in the initial stage. We note a_{K_1} and a_{K_3} form a diagonal of Q, and also a_{K_2} and a_{K_4} do. Each of computed values of directional stability radius is shown in Table I for some values of h together with those of ordinary stability radius (values in parentheses). We show how our algorithm works for these values of hs.

1) $h = 0.90$

Fig. 1 shows the result of algorithm 1 using the value of Table I at $h = 0.90$, with each node showing the partial problem. All 16 vertices are included by the union of $R(a_{K_j})$ from this figure, thus the interval polynomial (34) is stable at $h = 0.90$.

TABLE I
Computation Results of Directional Stability Radii [Ordinary Stability Radii $\rho(a_{K_i})$] for Some h

h	$\rho^*(a_{K_1})$	$\rho^*(a_{K_2})$	$\rho^*(a_{K_3})$	$\rho^*(a_{K_4})$
0.90	2.279	2.376	2.245	2.193
	(1.594)	(1.495)	(1.605)	(1.704)
0.95	1.360	1.378	1.237	1.303
	(0.871)	(0.625)	(0.966)	(1.200)
1.0	0.172	0.164	0.245	0.489
	(0.124)	(.0145)	(0.170)	(0.426)

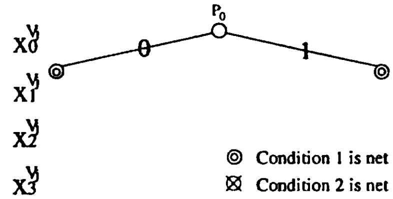

Fig. 1. Analysis result for $h = 0.9$.

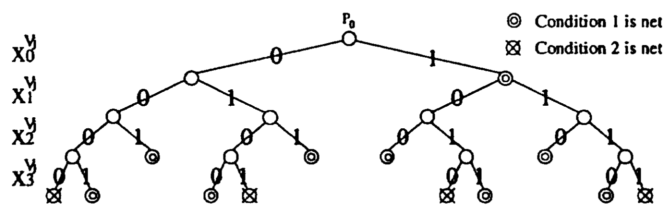

Fig. 2. Analysis result for $h = 0.95$.

2) $h = 0.95$

Fig. 2 shows the result of Algorithm 1 at $h = 0.95$. This figure indicates that the test continues to the final stage and that stability of four vertices are still unknown. We must check stability of these vertices one by one. As the results of Routh tests after coefficient transformations $f(s - h, a)$, one of them, $a = [88\ 216\ 181\ 49]^T$, is found to be unstable, that is, the interval polynomial (34) is unstable at $h = 0.95$.

3) $h = 1.0$

From Theorem 2, the fact $\rho^*(a_{K_2}) = 0.164 < 1.0 = \min_i |\beta_i - \alpha_i|$ indicates that the interval polynomial is unstable at $h = 1.0$.

Comparing Figs. 1 and 2, we find that, in case 1), the stability is analyzed more easily than in case 2) because of the following two points. The first is that we need less amount of computation for Algorithm 1, and the second is that no vertex polynomials have to be tested about their stability individually. These advantages comes from larger values of directional stability radii, indicating that the system has larger stability margin. In contrast, case 3) is the easiest among the three cases because applying algorithm 1 is unnecessary. The last case corresponds to a meager stability margin. In this way, a salient feature of the proposed method is in that the computational cost depends on the stability margin. In proportion to the margin, the cost could be notably reduced. On the other hand, too little margin could also conclude instability immediately. At worst, we are forced to depend on the point-wise tests.

To show the efficiency of our method, we measured computing time of stability analysis for a higher degree interval polynomial. For the case $n = 20$, the number of vertex polynomials is $2^{21} = 2\,097\,152$, therefore, it needs much time to check the stability of all the polynomials. With a Pentium compatible 600-MHz machine, computing time for the stability analysis of a 20th degree polynomial by Routh test after the coefficient transformation is 1650 (s). The analysis concludes the stability of the polynomial. The result of algorithm 1 also gives the same conclusion with computing time 223 (s), which shows that our method can remarkably reduce the computational burden in the stability analysis, in particular for higher degree polynomials.

V. Conclusion

As demonstrated, the directional stability radius is a useful stability analysis tool for parametrically uncertain systems represented by interval polynomials. Especially, our analysis method works efficiently for systems which have large stability margins. The directional stability radius can also find application in the stability analysis of delta-operator-induced interval systems. With a relation between stability of delta-operator-induced systems and shifted-Hurwitz stability in [7], the present method applies with no major modifications. For details, see [9]. Because of the nature of the directional stability radius, its use is not restricted to systems in which extreme point results hold. It could treat uncertain systems where only edge results are correct, or even worse those where no such convenience is available. In this sense, the authors believe the proposed methodology for stability analysis can cover much wider range of uncertain systems other than interval systems. Substantiating this line of studies is in progress.

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