

LETTER

Single-Parameter Characterizations of Schur Stability Property

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SUMMARY New equivalent characterizations are derived for Schur stability property of real polynomials. They involve a single scalar parameter, which can be regarded as a freedom incorporated in the given polynomials so long as the stability is concerned. Possible applications of the expressions are suggested to the latest results for stability robustness analysis in parameter space. Further, an extension of the characterizations is made to the matrix case, yielding one-parameter expressions of Schur matrices.

key words: Schur polynomial, Schur matrix, characterizations, robustness, free parameter

1. Introduction

Stability of linear systems is, in general, robust in the sense that the property is maintained in the presence of small perturbations of system parameters. Schur stability, an important notion in digital signal processing and analysis and design of discrete-time systems, is not an exception. For instance, given a Schur polynomial, it can safely be said that a set of polynomials which are enough contiguous to the nominal one are also Schur stable. In association with stability robustness analysis in parameter space, one is often concerned with the shape and size of such a “vicinity” where the stability is ensured. To comprehend further the stability property in parameter space, such concepts as stability direction or stability radius have been proposed, and discussed for Schur stability and Hurwitz stability alike [8], [9]. Still, a complete picture of stability regions even in polynomial coefficient space or matrix entry space is a way off except for lower order cases.

This paper contributes to understanding the Schur stability property in these spaces by deriving some new equivalent characterizations of the property that fit in the latest stability robustness analysis results. The purpose of this paper is not to obtain another Schur stability test but to show one-parameter characterizations, which enables to broaden the scope of parameter space robust stability. The paper is organized as follows.

Next section provides the main result on equivalent characterizations of Schur polynomials along with its proof. Section 3 is devoted to their possible applications including implications in stability direction and stability radius issues. An extension to the matrix case is also made in this section. Finally, conclusions are drawn in Sect. 4.

2. Main Result and Its Proof

Consider a real n th-degree polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_0 > 0. \quad (1)$$

Schur stability of (1) implies and is implied by the existence of its n zeros inside the unit disk on the complex plane. We start with stating the main result of this section.

Theorem 1: The following statements are equivalent.

- i) $f(z)$ is Schur stable.
- ii) For any real γ ($|\gamma| < 1$), $f(z) + \gamma z^n f(\frac{1}{z})$ is Schur stable.
- iii) For some real γ ($|\gamma| < 1$), $f(z) + \gamma z^n f(\frac{1}{z})$ is Schur stable.
- iv) For any real δ ($|\delta| > 1$), $\delta f(z) + z^n f(\frac{1}{z})$ is Schur stable.
- v) For some real δ ($|\delta| > 1$), $\delta f(z) + z^n f(\frac{1}{z})$ is Schur stable.

This theorem can be proven by combining the two preliminary lemmas shown below.

Lemma 1 [6], [7]: A necessary and sufficient condition for $f(z)$ to be Schur stable is that the zeros of $f_m(z)$ and those of $f_a(z)$ are all simple and are located alternately on the unit circle and that $|a_n/a_0| < 1$ holds. Here, $f_m(z)$ and $f_a(z)$ are mirror component and antimirror component of $f(z)$ defined by $f_m(z) = \frac{1}{2}(f(z) + z^n f(\frac{1}{z}))$ and $f_a(z) = \frac{1}{2}(f(z) - z^n f(\frac{1}{z}))$, respectively.

This lemma is known as the discrete-time version of Hermite Biehler theorem. Note that $f(z) = f_m(z) + f_a(z)$.

Lemma 2: For the scalar function with two real arguments,

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$$r(\lambda, k) := \frac{1 + k - \lambda(1 - k)}{1 + k + \lambda(1 - k)}, \quad (2)$$

we have;

- i) for $\lambda > 0$, $|r(\lambda, k)| < 1 \Leftrightarrow |k| < 1$
- ii) for $|k| < 1$, $|r(\lambda, k)| < 1 \Leftrightarrow \lambda > 0$.

Statement i) assumes that λ and k are a parameter and a variable, respectively, while ii) does that k and λ are so, respectively.

The proof of the above lemma is elementary and thus omitted. Now we are in position to prove Theorem 1.

Proof of Theorem 1:

i) \rightarrow ii) Assume $f(z)$ is Schur stable. Lemma 1 says that the zeros of $f_m(z)$ and $f_a(z)$ are all simple and interlacing on the unit circle and furthermore that $|a_n/a_0| < 1$ is satisfied. Consider a polynomial $f_m(z) + \alpha f_a(z)$ with α being an arbitrary positive number. We will prove Schur stability of $f_m(z) + \alpha f_a(z)$. We have $f_m(z) + \alpha f_a(z) = \frac{1}{2}\{(1+\alpha)f(z) + (1-\alpha)z^n f(\frac{1}{z})\}$, whose mirror polynomial and antimirror polynomial are given by $f_m(z)$ and $\alpha f_a(z)$, respectively. By the assumption, therefore, the requirements for simple zeros and for interlacing properties are fulfilled for $f_m(z) + \alpha f_a(z)$. It remains to be proven that the modulus of the ratio of the lowest-degree coefficient to the highest one is less than unity. The ratio is given by

$$\frac{a_0 + a_n - \alpha(a_0 - a_n)}{a_0 + a_n + \alpha(a_0 - a_n)} = r(\alpha, k), \quad k = \frac{a_n}{a_0}. \quad (3)$$

Since $|k| < 1$ by Lemma 1, Lemma 2 ii) immediately gives $|r(\alpha, k)| < 1$, which concludes Schur stability of $f_m(z) + \alpha f_a(z)$ due again to Lemma 1. To arrive at ii), we notice that $f_m + \alpha f_a(z) = \frac{1}{2}(1+\alpha)(f(z) + \gamma z^n f(\frac{1}{z}))$ with $\gamma = (1-\alpha)/(1+\alpha)$, which is a bijective mapping from $\alpha > 0$ to $|\gamma| < 1$. This demonstrates the claim.

ii) \rightarrow iii) Obvious.

iii) \rightarrow i) We assume that $f(z) + \gamma z^n f(\frac{1}{z})$ is stable for some γ ($|\gamma| < 1$). The above arguments confirm that corresponding to γ there exists an $\alpha > 0$ such that simple zero and interlacing properties are satisfied for $f_m(z) + \alpha f_a(z)$. These properties remain in force, even α is set to be unity. We have now only to show that for $r(\alpha, k)$ in (3) with some $\alpha > 0$ $|r(\alpha, k)| < 1$ implies $|k| < 1$. This comes easily from i) in Lemma 2. Now, Lemma 1 ensures Schur stability of $f_m(z) + f_a(z) = f(z)$.

The equivalences of iv), v) and i) are proven through i) \rightarrow iv) \rightarrow v) \rightarrow i) in the same way as above, by considering the polynomial $\beta f_m(z) + f_a(z) = \frac{1}{2}\{(\beta+1)f(z) + (\beta-1)z^n f(\frac{1}{z})\} = \frac{\beta-1}{2}(\delta f(z) + z^n f(\frac{1}{z}))$ with $\delta = (\beta+1)/(\beta-1)$ and $\beta > 0$, along with the ratio,

$$\frac{(a_0 + a_n) - \frac{1}{\beta}(a_0 - a_n)}{(a_0 + a_n) + \frac{1}{\beta}(a_0 - a_n)} = r(\frac{1}{\beta}, k). \quad (4)$$

This completes the proof of Theorem 1. \square

Statements iv) and v) are nothing but the consequences of ii) and iii) when γ is set to $1/\delta$ formally. But, we listed them in order, for one thing, to make clear the behavior of the zeros as $\delta \rightarrow \pm\infty$ and, for another, to facilitate the discussions given later on instability.

The authors were informed by one of the reviewers that Theorem 1 would be verifiable by means of Rouché's theorem as well along the line of [18]. An alternative way is thus possible to the obtained characterizations.

3. Possible Applications and an Extension

3.1 Possible Applications of the Obtained Characterizations

In this subsection, several possible applications of the expressions obtained in the previous section are suggested. Some of them illuminate the implications which the characterizations bring into the notions in the latest parametric stability analysis. We will not go into detail of each of the applications, because it is beyond the scope of the paper.

The first possible aspect where Theorem 1 can find its effectiveness is the improvement of sufficient stability conditions. So far as Schur stability of polynomials are concerned, a variety of sufficient conditions are available (see, for example, [3]). The equivalence i) \leftrightarrow iii) in Theorem 1 could contribute to improving these conditions by introduction of the free parameter. This can be done as follows. Take any of the sufficient conditions, apply it not to $f(z)$ but to $f(z) + \gamma z^n f(\frac{1}{z})$ and choose γ so that the condition is fulfilled. Since $\gamma = 0$ corresponds to the original condition, the condition with γ may enlarge the possibility that it is met by given coefficients. We illustrate an example. The condition,

$$\frac{1}{\omega} + \max_{1 \leq k \leq n} \{|a_k| \omega^{k-1}\} < 1, \quad \exists \omega > 1 \quad (5)$$

ensures Schur stability of (1) with $a_0 = 1$ [3]. Applying (5) to the polynomial, $f(z) + \gamma z^n f(\frac{1}{z}) = (a_0 + \gamma a_n)z^n + (a_1 + \gamma a_{n-1})z^{n-1} + \dots + (a_n + \gamma a_0)$, we have

$$\max_{1 \leq k \leq n} \{|a_k + \gamma a_{n-k}| \omega^k\} < (\omega - 1)(1 + \gamma a_n) \quad (6)$$

with $\omega > 1$ and $|\gamma| < 1$. If we can find a parameter pair (ω, γ) such that (6) is satisfied, then Schur stability of (1) follows. In [10], some other examples where the improvement is achieved by the same scenario are indicated. Apropos, this scheme is, of course, no more in force for any of exact Schur stability criteria.

The second potential application, however, is concerning a very stability condition which is exact, Jury test [1]. We can, in fact, derive the Jury test using Theorem 1. In this case, we use the equivalences i)

\leftrightarrow ii) \leftrightarrow iii). Selecting γ as $\gamma = -\frac{a_n}{a_0}$, we see that the constant term in $f(z) + \gamma z^n f(\frac{1}{z})$ drops, that means one of its zeros is moved to the origin. By the equivalent relations, Schur stability of $f(z)$ is identical to that of the polynomial where this zero is removed, that is, $\{f(z) - \frac{a_n}{a_0} z^n f(\frac{1}{z})\}/z$. We repeat the procedure for the resultant $(n-1)$ th degree polynomial. In this way, one can reduce the degree of polynomials one by one, keeping the stability property intact, to reach the zero degree one. The Jury stability test is no other than the conditions that the modulus of γ selected in each step of the above process is less than unity. Another proof of the Jury test has also been given in [5] recently, where the root locus technique is employed. The authors of [5] assert that students and engineers are forced to accept the test method on faith without being exposed to its proof. This may be true worldwide and hints the need to provide at least a clue or a tip on how it can be derived from some acceptable standard results.

Thirdly, Theorem 1 can give some insight into such stability robustness concepts in coefficient space $\mathbf{a} := (a_0, a_1, \dots, a_n)^t$, (t) :transpose, as the stability direction and the stability radius [8], [9]. We begin with the stability (convex) direction, which is the direction in coefficient space corresponding to the difference polynomial $f_2(z) - f_1(z)$ where $f_1(z)$ and $f_2(z)$ are any stable polynomials of the same degree and their convex combinations are also stable. An exact condition on the stability direction is obtained for both Hurwitz and Schur cases [8], [11]. As it turns out, however, the notion is quite restrictive because of the arbitrariness of $f_1(z)$ and $f_2(z)$, and an attempt is made to weaken the requirement by fixing the pivot polynomial $f_1(z)$. This weak concept is called local stability direction and considered for the Hurwitz case [12]. Unfortunately, its Schur counterpart is not yet available. Now, with these definitions and results in mind, one could easily see that Theorem 1 i)–ii) give local Schur stability directions, $\pm(f(z) - z^n f(\frac{1}{z}))$ or $\pm f_a(z)$ with the pivot $f(z)$. It is interesting that the directions are obtainable in an analytical form for every Schur polynomial. In the same vein, the equivalences iv)–v) show the instability counterpart. Namely, $\pm f_m(z)$ is local Schur instability directions with the pivot $z^n f(\frac{1}{z})$, which is anti-Schur stable (i.e. all the zeros are outside the unit circle). In this way, Theorem 1 can give a certain amount of information on the stability directions. We next pass to the stability radius R_f for the polynomial (1) under coefficient perturbations $\Delta \mathbf{a} := (\Delta a_0, \Delta a_1, \dots, \Delta a_n)^t$. With some appropriate vector norm $\|\cdot\|$, R_f for a nominal Schur polynomial (1) can be defined as

$$R_f := \min\{\|\Delta \mathbf{a}\| \mid \sigma(f(z) + \Delta f(z)) \cap e^{j\omega} \neq \emptyset, \omega \in [-\pi, \pi]\} \quad (7)$$

where $\sigma(\cdot)$ denotes the set of the zeros of polynomials and $f(z) + \Delta f(z) = (a_0 + \Delta a_0)z^n + (a_1 + \Delta a_1)z^{n-1} +$

$\dots + (a_n + \Delta a_n)$. To obtain R_f , we have to pay a considerable price, because an optimization procedure is needed [9]. Now, we readily observe that i)–iii) in Theorem 1 give an upper bound for R_f as

$$R_f \leq \|\bar{\mathbf{a}}\|, \quad \bar{\mathbf{a}} := (a_n, a_{n-1}, \dots, a_0)^t \quad (8)$$

by regarding $z^n f(\frac{1}{z})$ as $\Delta f(z)$. The bound (8) works for any Schur polynomial and can be used as an initial guess in the optimization process. Theorem 1 iv)–v) can likewise give a bound for the instability radius $R_{\bar{f}}$ around $\bar{\mathbf{a}}$, where any polynomial $z^n f(\frac{1}{z}) + \Delta f(z)$ with $\|\Delta \mathbf{a}\| \leq R_{\bar{f}}$ is Schur unstable.

3.2 An Extension to the Matrix Case

In this subsection, the foregoing results are extended to the matrix case so that some characterizations of Schur matrices are correct. The obtained one-parameter characterizations can also give insight into the stability direction and stability radius in matrix entry space. The discussions of this subsection are based on Theorem 1 and the companion form defined by

$$L_A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{bmatrix} \quad (9)$$

which corresponds to the characteristic polynomial (1) with $a_0 = 1$ associated with A . The special $n \times n$ matrix given below also plays a role.

$$J := \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \quad (10)$$

Now, let T be a real nonsingular matrix which brings a given matrix A to the companion form L_A , that is, $TAT^{-1} = L_A$. The main result of this section is stated as follows.

Theorem 2: Let A be nonsingular. Then, we have equivalent statements:

- i) A is Schur stable.
- ii) For any real γ ($|\gamma| < 1$), $A + g(\gamma)C_A$ is Schur stable.
- iii) For some real γ ($|\gamma| < 1$), $A + g(\gamma)C_A$ is Schur stable.

Here, C_A is given by

$$C_A = \bar{T}A^{-1}\bar{T}, \quad \bar{T} = T^{-1}JT. \quad (11)$$

and $g(\gamma)$ by

$$g(\gamma) = \frac{\gamma d}{1 + \gamma d}, \quad d = \det(-A). \quad (12)$$

Remark : Note that $d = a_n$ and therefore $|d| < 1$. The function $g(\gamma)$, which maps $|\gamma| < 1$ to the interval (\underline{g}, \bar{g}) with $\underline{g} = -|d|/(1 - |d|)$ and $\bar{g} = |d|/(1 + |d|)$, is monotonic and satisfies $|g(\gamma)| < \bar{g}$, $\forall |\gamma| < 1$.

Since the proof of Theorem 2 is just a combination of Theorem 1 and the above companion form as mentioned, we leave it to the interested readership.

Theorem 2 enables us to develop parallel discussions to those given in the previous subsection on the stability direction and stability radius of polynomials [14]–[17], shedding some light on stability robustness issues in matrix entry space.

4. Concluding Remarks

New equivalent characterizations of Schur stability property are derived for real polynomials. They include a single parameter, which makes them possible to find applications to stability and robust stability problems of dynamical systems. Several among them are outlined and suggested. An extension to the matrix case is also made to yield equivalent parameterized expressions of Schur matrices. They could give some insight into stability robustness problems for state space models as well.

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