LETTER

New Formulas on Orthogonal Functionals of Stochastic Binary Sequence with Unequal Probability

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SUMMARY This paper deals with an orthogonal functional expansion of a non-linear stochastic functional of a stationary binary sequence taking ± 1 with unequal probability. Several mathematical formulas, such as multivariate orthogonal polynomials, recurrence formula and generating function, are given in explicit form. A formula of an orthogonal functional expansion for a stochastic functional is presented; the completeness of expansion is discussed in Appendix.

key words: *stochastic binary sequence, orthogonal stochastic functional, functional expansion, multivariate orthogonal polynomials*

1. Introduction

This paper deals with an orthogonal functional expansion of a stochastic functional of a stationary binary sequence. Such a sequence is a mathematical model of digital signals in communication engineering, but it has important applications to physical problems.

Recently, waves in binary random media and scattering from binary rough surfaces have received much interest, because the Anderson localization [1],[2] in such media becomes a physical principle of new optical devices [3], and because digital data are stored by surface deformations called pits in recording devices such as a compact disk. In principle, these binary random media and rough surfaces may be modeled by a stationary binary sequence, which takes two values with unequal probability in many cases. Since the wave field in such a medium may be considered as a stochastic functional of a binary sequence, we need a systematic theory of a stochastic functional of binary sequences to represent the wave field.

A theory of a stochastic functional of binary sequences was first introduced by Ogura [4] as a close analogy to the Wiener-Hermite expansion [5], where a method of constructing multivariate orthogonal polynomials and mathematical definition of orthogonal stochastic functionals were discussed but no explicit formulas were given. On the other hand, Aracil [6] and others [7] studied a stochastic functional of binary and ternary signals as a method of nonlinear system identification. In a previous paper [8], however, we gave several explicit formulas on the multivariate orthogonal polynomials associated with a binary sequence, which were successfully applied to the wave scattering from a binary rough surface [9].

In these works [4],[6]–[9], however, discussions were restricted to a special case where a binary sequence takes two values with equal probability. Removing such restriction, we obtain an explicit formula of the multivariate orthogonal polynomials, recurrence formula and generating function for an unequal probability case. We present a formula of an orthogonal functional expansion for a stochastic functional, the completeness of which is discussed in Appendix. We also discuss an example of orthogonal functional expansion and a stationary sequence generated by the stationary binary sequence.

2. Multivariate Orthogonal Polynomials

Let $\{b_i, i = 0, \pm 1, \pm 2, \cdots\}$ be an independent stationary binary sequence taking ± 1 with unequal probability

$$P(b_i = 1) = \frac{1+\mu}{2}, \qquad P(b_i = -1) = \frac{1-\mu}{2}, \qquad (1)$$

where real μ is the average parameter with $|\mu| < 1$. One easily finds the average and correlation

$$\langle b_i \rangle = \mu, \quad \langle (b_i - \mu)(b_j - \mu) \rangle = (1-\mu^2)\delta(i,j), \quad (2)$$

where the angle brackets denote ensemble average and $\delta(i,j)$ is the Kronecker delta. If $\{i_1, i_2, \cdots, i_m\}$ is a distinct set of integers, $b_{i_1}, b_{i_2}, \cdots, b_{i_m}$ are mutually independent, so that one easily finds

$$\langle (b_{i_1} - \mu)(b_{i_2} - \mu) \cdots (b_{i_m} - \mu) \rangle = 0. \qquad (3)$$

When $\{i_1, i_2, \cdots, i_m\}$ and $\{j_1, j_2, \cdots, j_n\}$ are two distinct sets of integers, we get an orthogonality relation

$$\begin{aligned} &\langle (b_{i_1} - \mu)(b_{i_2} - \mu) \cdots (b_{i_m} - \mu) \\ &\quad \cdot (b_{j_1} - \mu)(b_{j_2} - \mu) \cdots (b_{j_n} - \mu) \rangle \\ &= (1-\mu^2)^n \delta(m,n) \delta^{(m)}(i,j), \end{aligned} \qquad (4)$$

$$\begin{aligned} \delta^{(m)}(i,j) = &\sum_{l_1=1}^{m} \delta(i_1, j_{l_1}) \sum_{\substack{l_2=1 \\ l_2 \neq l_1}}^{m} \delta(i_2, j_{l_2}) \\ &\times \cdots \times \sum_{\substack{l_m=1 \\ l_m \neq l_1, l_2, \cdots, l_{m-1}}}^{m} \delta(i_m, j_{l_m}), \end{aligned} \qquad (5)$$

Manuscript received March 17, 1997.
Manuscript revised September 29, 1997.

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where $\delta^{(m)}(i,j)$ involves $m!$ terms. The relation (4) may be proved in terms of the moment generating function [8].

Let us define multivariate orthogonal polynomials $\{\hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_{m-1}}, b_{i_m}), m = 0, 1, 2, \cdots\}$ associated with the binary sequence. However, it should be noted that, for example, $b_i b_j$ is a second order polynomial for $i \neq j$, but becomes a constant $b_i b_j = 1$ when $i = j$. Taking this fact in mind, we introduce a binary function $\Delta_m(i_1, i_2, \cdots, i_m)$ by

$$\begin{aligned}\Delta_m(i_1, i_2, \cdots, i_m) &= \left[1 - \sum_{k=1}^{m-1} \delta(i_m, i_k)\right]\left[1 - \sum_{k=1}^{m-2} \delta(i_{m-1}, i_k)\right] \\ &\times \cdots \times [1 - \delta(i_3, i_2) - \delta(i_3, i_1)][1 - \delta(i_2, i_1)] \\ &= \begin{cases} 1, & (i_1, i_2, \cdots, i_m \text{ are all distinct}) \\ 0, & (\text{ any other case }) \end{cases}, \end{aligned} \tag{6}$$

which vanishes for m-dimensional diagonal arguments. Then, we define the multivariate orthogonal polynomials by

$$\begin{aligned} &\hat{B}_0 = 1, \quad \hat{B}_1(b_i) = b_i - \mu, \\ &\hat{B}_2(b_i, b_j) = [1 - \delta(i,j)](b_i - \mu)(b_j - \mu), \\ &\cdots \quad \cdots \quad \cdots \\ &\hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_m}) = \Delta_m(i_1, i_2, \cdots, i_m) \\ &\quad \times (b_{i_1} - \mu)(b_{i_2} - \mu) \cdots (b_{i_m} - \mu), \end{aligned} \tag{7}$$

which are symmetrical with respect to their variables. However, these polynomials may be obtained from a generating function Q as

$$\hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_m}) = \frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2}} \cdots \frac{\partial}{\partial t_{i_m}} Q \Big|_{t_0 = t_{\pm 1} = t_{\pm 2} = \cdots = 0}. \tag{8}$$

The generating function Q is formally given as

$$Q = \prod_{k=-\infty}^{\infty} [1 + (b_k - \mu) \cdot t_k], \tag{9}$$

which may be derived later. Since $\partial t_m / \partial t_l = \delta(l, m)$, one easily gets (7) from (8). Manipulating (7) and (6), we find the recurrence formula,

$$\begin{aligned} &\hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_m}) \\ &= \hat{B}_{m-1}(b_{i_1}, b_{i_2}, \cdots, b_{i_{m-1}}) \hat{B}_1(b_{i_m}) \\ &+ 2\mu \sum_{l=1}^{m-1} \hat{B}_{m-1}(b_{i_1}, b_{i_2}, \cdots, b_{i_{m-1}}) \delta(i_m, i_l) \\ &- (1 - \mu^2) \sum_{l=1}^{m-1} \hat{B}_{m-2}(b_{i_1}, \cdots, b_{i_{l-1}}, b_{i_{l+1}}, \cdots, b_{i_{m-1}}) \\ &\times \delta(i_m, i_l) + (1 - \mu^2) \sum_{l=1}^{m-1} \sum_{\substack{k=1 \\ k \neq l}}^{m-1} \delta(i_m, i_l) \delta(i_k, i_l) \\ &\times \hat{B}_{m-2}(b_{i_1}, \cdots, b_{i_{l-1}}, b_{i_{l+1}}, \cdots, b_{i_{m-1}}). \end{aligned} \tag{10}$$

Using (3), (4) and (7), we easily find averages and orthogonality relation of the multivariate polynomials,

$$\langle \hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_{m-1}}, b_{i_m}) \rangle = \delta(m, 0), \tag{11}$$

$$\begin{aligned} &\langle \hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_m}) \hat{B}_n(b_{j_1}, b_{j_2}, \cdots, b_{j_n}) \rangle \\ &= (1 - \mu^2)^n \Delta_m(i_1, \cdots, i_m) \Delta_n(j_1, \cdots, j_n) \\ &\quad \times \delta(m, n) \delta^{(m)}(i, j), \end{aligned} \tag{12}$$

which holds for any non-negative integers m and n. We note that (7)–(10) are new formulas obtained in this paper. When $\mu = 0$ and the binary sequence takes ± 1 with equal probability, these formulas are reduced to formulas in Reference [8].

3. Orthogonal Functional Expansion

We assume a sample point ω is an infinitely dimensional vector

$$\omega = (\cdots, b_{-2}, b_{-1}, b_0, b_1, \cdots) \qquad \omega_i = b_i, \tag{13}$$

where ω_i is the i-th component of ω. Also, we assume that the sample space Ω is made up of all such sample points. Then, any function of ω implies a functional of the binary sequence $\{b_i\}$. By $P(\omega)$ and $L^2(\Omega, P)$, we denote the probability measure [10] on Ω and the class of functions $g(\omega)$ with

$$||g(\omega)||^2 = \langle |g(\omega)|^2 \rangle < \infty, \tag{14}$$

respectively. Here, $|| \bullet ||$ denotes norm.

For $g(\omega) \in L^2(\Omega, P)$, we have an orthogonal functional expansion

$$\begin{aligned} g(\omega) &= g_0 + \sum_{k=-\infty}^{\infty} g_1(k) \hat{B}_1(b_k) + \cdots \\ &+ \sum_{i_1, i_2, \cdots, i_m = -\infty}^{\infty} g_m(i_1, i_2, \cdots, i_m) \hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_m}) \\ &+ \cdots, \end{aligned} \tag{15}$$

which holds in ensemble mean square sense and may be proved in Appendix. Because the binary function $\Delta(i_1, i_2, \cdots, i_m)$ and the multivariate polynomial $\hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_m})$ vanish for m-dimensional diagonal arguments, the diagonal components of the kernel functions are indefinite. Because of the symmetry of multivariate polynomials, the kernel functions may not be determined uniquely. However, we always assume that the kernel functions have zero-diagonal components and are symmetrical with respect to their arguments. Then, such zero-diagonal symmetrical kernel functions are obtained uniquely from (6), (12) and (15) as

$$g_m(i_1, \cdots, i_m) = \frac{\langle g(\omega) \hat{B}_m(b_{i_1}, \cdots, b_{i_m}) \rangle}{(1 - \mu^2)^m m!}, \quad m = 0, \pm 1, \pm 2, \cdots. \tag{16}$$

Since (15) is complete, one gets the Parseval relation,

$$\langle|g(\omega)|^2\rangle = \sum_{m=0}^{\infty}(1-\mu^2)^m m! \times \sum_{i_1,i_2,\cdots,i_m=-\infty}^{\infty} |g_m(i_1,i_2,\cdots,i_m)|^2. \quad (17)$$

As a simple example, let us consider the orthogonal functional expansion of

$$g(\omega) = \exp\left(\sum_{k=-\infty}^{\infty} f_1(k)b_k\right). \quad (18)$$

If $\sum_{k=-\infty}^{\infty}\left|Re[f_1(k)]\right| < \infty$, Re being the real part, $g(\omega)$ belongs to $L^2(\Omega, P)$. Equation (18) is regarded as the output of a cascade system involving a linear filter and an exponential nonlinear for the input of binary sequence. However, this example gives the generating function Q for $\hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_m})$. By (16) and (18), we may calculate the kernel functions

$$g_0 = \prod_{k=-\infty}^{\infty}\{\cosh[f_1(k)] + \mu\sinh[f_1(k)]\},$$
$$g_m(i_1,i_2,\cdots,i_m) = \frac{g_0\Delta_m(i_1,\cdots,i_m)}{m!}t_{i_1}t_{i_2}\cdots t_{i_m},$$
$$t_k = \frac{\tanh[f_1(k)]}{1+\mu\tanh[f_1(k)]}. \quad (19)$$

Substituting the result (19) into (15), we get the orthogonal functional expansion,

$$g(\omega) = g_0 + \frac{g_0}{1!}\sum_k t_k\hat{B}_1(b_k) + \frac{g_0}{2!}\sum_{k,l} t_k t_l \hat{B}_2(b_k, b_l) + \frac{g_0}{3!}\sum_{i,j,k} t_i t_j t_k \hat{B}_3(b_i, b_j, b_k) + \cdots. \quad (20)$$

On the other hand, this expansion can be derived by another method. For $b_k = \pm 1$, we find an identity

$$e^{f_1(k)b_k} = \{\cosh[f_1(k)] + \mu\sinh[f_1(k)]\}[1 + (b_k - \mu)t_k],$$

where t_k is given by (19). Then we have

$$\begin{aligned} g(\omega) &= g_0 \prod_{k=-\infty}^{\infty}[1 + t_k(b_k - \mu)] \\ &= g_0 + g_0\sum_k t_k(b_k - \mu) \\ &\quad + g_0\sum_{k<l} t_k t_l (b_k - \mu)(b_l - \mu) \\ &\quad + g_0\sum_{i<j<k} t_i t_j t_k (b_i - \mu)(b_j - \mu)(b_k - \mu) + \cdots, \end{aligned} \quad (21)$$

which is the same expansion as (20). Equating (20) to (21), we find the generating function Q in (9).

4. Stationary Sequence

We have considered the functional expansion of a stochastic functional $g(\omega)$, which is a random variable. This section discusses a stationary sequence generated by the stationary binary sequence.

Let us define the shift from a sample point ω to another sample point $T\omega$ in the sample space Ω by

$$T \cdot b_i = b_{i+1}, \quad T\omega = (\cdots, b_{-1}, b_0, b_1, b_2, \cdots). \quad (22)$$

Because $\{b_i\}$ is an independent stationary sequence [11], if a sample sequence ω exists with $P(\omega)$, the shifted sequence $T\omega$ must exist with the same probability $P(T\omega) = P(\omega)$, which means measure-preserving. Thus, the shift T is a measure-preserving transformation in Ω with properties: $T^0 = 1$(identity); $T^m T^n = T^{m+n}$, where m and n are any integers. Furthermore, for any random variable $g(\omega)$, $g(T^n\omega)$ is a stationary sequence and becomes ergodic [10],[11]. Applying the shift to the multivariate polynomials, we obtain

$$\begin{aligned} &T^\beta \hat{B}_m(b_{i_1}, b_{i_2}, \cdots, b_{i_{m-1}}, b_{i_m}) \\ &\quad = \hat{B}_m(b_{i_1+\beta}, b_{i_2+\beta}, \cdots, b_{i_{m-1}+\beta}, b_{i_m+\beta}), \end{aligned} \quad (23)$$

where β is any integer. Using (23) and (15), we obtain the functional expansion for $g(T^\beta\omega)$

$$g(T^\beta\omega) = g_0 + \sum_{k=\infty}^{\infty} g_1(k-\beta)\cdot\hat{B}_1(b_k) + \sum_{k,l=\infty}^{\infty} g_2(k-\beta, l-\beta)\cdot\hat{B}_2(b_k, b_l) + \cdots, \quad (24)$$

which is a stationary sequence generated from the binary sequence $\{b_i\}$. The correlation function may be easily calculated as

$$\begin{aligned} &\langle g(T^\beta\omega)g^*(\omega)\rangle \\ &\quad = |g_0|^2 + 1!(1-\mu^2)\sum_{k=\infty}^{\infty} g_1(k-\beta)g_1^*(k) \\ &\quad\quad + 2!(1-\mu^2)^2\sum_{k \neq l} g_2(k-\beta, l-\beta)g_2^*(k,l) + \cdots, \end{aligned} \quad (25)$$

where asterisk means the complex conjugate. Note that any statistical properties of $g(T^\beta\omega)$ can be obtained from one sample sequence, because $g(T^\beta\omega)$ is ergodic.

5. Conclusion

By formal discussions, we have obtained new formulas concerning the multivariate orthogonal polynomials of a stationary binary sequence taking ± 1 with unequal probability. Further, we have obtained a formula on orthogonal functional expansion of a stochastic functional. Since these formulas are given in explicit form, they can be applied to various problems in random theory, such as wave scattering from random media and rough surfaces, and non-linear system identification using the input of binary sequence. However, these applications will be left for future study.

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Appendix: Functional Expansion

This appendix briefly describes a proof of the functional expansion (15). However, we start with a special case, where $g(\omega) = g^{(2M+1)}(\omega)$ is a step function of special type such that $g^{(2M+1)}(\omega)$ depends on only the $2M+1$ variables: $b_{-M}, b_{-M+1}, \cdots, b_0, b_1, \cdots, b_M$, and is independent of other variables. Such a function is represented in terms of the multivariate polynomials as [10]

$$\begin{aligned} g^{(2M+1)}(\omega) &= g^{(2M+1)}(b_{-M}, b_{-M+1}, \cdots, b_M) \\ &= g_0 + \sum_{k_1=-M}^{M} g_1(k_1)\hat{B}_1(b_{k_1}) \\ &\quad + \sum_{k_1,k_2=-M}^{M} g_2(k_1,k_2)\hat{B}_2(b_{k_1}, b_{k_2}) + \cdots \\ &\quad + \sum_{k_1,k_2,\cdots,k_{2M+1}=-M}^{M} g_{2M+1}(k_1,k_2,\cdots,k_{2M+1}) \\ &\quad \times \hat{B}_{2M+1}(b_{k_1}, b_{k_2}, \cdots, k_{2M+1}), \end{aligned} \quad (A\cdot 1)$$

which should be understood as a special case of (15). Here, g_0, $g_1(-M)$, $g_1(-M+1), \cdots, g_1(M)$, $g_2(-M,-M), \cdots, g_{2M+1}(M,M,\cdots,M)$ are coefficients, the number of which is reduced to 2^{2M+1}, because $g_m(k_1,k_2,\cdots,k_m)$ is symmetrical and zero-diagonal. These coefficients are uniquely determined from $g^{(2M+1)}(b_{-M}, b_{-M+1}, \cdots, b_M)$ when b_k's take ± 1 independently. Therefore, (A·1) holds for each vector $(b_{-M}, b_{-M+1}, \cdots, b_M)$ and hence it is valid in $L^2(\Omega, P)$ sense. Thus, we have proved (15) for a step function of special type.

Let us prove (15) for a general case. First, we introduce a transformation

$$x^{(2M+1)} = \sum_{k=0}^{M} \frac{1+b_k}{2} 2^{-2k-1} + \sum_{k=1}^{M} \frac{1+b_{-k}}{2} 2^{-2k}. \quad (A\cdot 2)$$

Here, $(1+b_k)/2$ is a bit and $x^{(2M+1)}$ is a $(2M+1)$-bit binary number, which runs over $\{m/2^{2M+1}, m = 0,1,2,\cdots,2^{2M+1}-1\}$. When M goes to infinity, ω and Ω are transformed into a real number $x = x^{(\infty)}$ and the interval $\Omega_R = [0,1]$. Also, $P(\omega)$, $g(\omega)$ and $g^{(2M+1)}(\omega)$ are transformed into $\tilde{P}(x)$, $\tilde{g}(x)$ and $\tilde{g}^{(2M+1)}(x)$ on the interval Ω_R, respectively. Here, $\tilde{g}^{(2M+1)}(x)$ is constant in an interval: $m/2^{2M+1} \leqq x < (m+1)/2^{2M+1}$, $m = 0,1,2,\cdots,2^{2M+1}-1$. Also, $L^2(\Omega,P)$ is transformed into $L^2(\Omega_R, \tilde{P})$ with $||g(\omega)|| = ||\tilde{g}(x)||$. Obviously, there is one-by-one correspondence between $\tilde{g}(x)$ and $g(\omega)$ in ensemble mean square sense.

It is known that any function $\tilde{g}(x)$ belonging to $L^2(\Omega_R, \tilde{P})$ is approximated arbitrarily closely in $L^2(\Omega_R, \tilde{P})$ sense by a step function $\tilde{g}^{(N)}(x)$, which may be written as [5],[11]

$$\begin{aligned} &0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \\ &\tilde{g}^{(N)}(x) = \tilde{g}_n^{(N)}, \quad x_n \leqq x < x_{n+1}. \end{aligned} \quad (A\cdot 3)$$

Thus, for any $\epsilon > 0$, there exists $\tilde{g}^{(N)}(x)$ such that

$$||\tilde{g}(x) - \tilde{g}^{(N)}(x)|| < \epsilon. \quad (A\cdot 4)$$

On the other hand, such a step function $\tilde{g}^{(N)}(x)$ is approximated arbitrarily closely by a step function of special type $\tilde{g}^{(2M+1)}(x)$. Thus, there is $\tilde{g}^{(2M+1)}(x)$ satisfying

$$||\tilde{g}^{(N)}(x) - \tilde{g}^{(2M+1)}(x)|| < \epsilon. \quad (A\cdot 5)$$

This is obvious when $\tilde{g}^{(N)}(x)$ and $\tilde{g}^{(2M+1)}(x)$ are functions on the real axis. From (A·4) and (A·5), we obtain

$$\begin{aligned} ||\tilde{g}(x) - \tilde{g}^{(2M+1)}(x)|| &\leqq ||\tilde{g}(x) - \tilde{g}^{(N)}(x)|| \\ &+ ||\tilde{g}^{(N)}(x) - \tilde{g}^{(2M+1)}(x)|| < 2\epsilon. \end{aligned} \quad (A\cdot 6)$$

Since ϵ is arbitrary, this means that

$$\lim_{M\to\infty} ||g(\omega) - g^{(2M+1)}(\omega)|| = 0. \quad (A\cdot 7)$$

Therefore, the functional expansion (15) is complete in $L^2(\Omega, P)$ sense and the Parseval relation (17) holds.