Branching rules for $SO(n+3)/SO(3)\times SO(n)$

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1. INTRODUCTION.

In the study of geometry of a homogeneous space G/K, where G is a compact connected Lie group and K is its closed subgroup, we sometimes need to consider how an irreducible G-module decomposes into irreducible K-modules when the action of G is restricted to that of K. The rule which governs the decomposition is called the branching rule for G/K. Moreover, as is seen in the example of [1], we usually need that the rule should be given as to clarify all the irreducible G-modules that include a specified irreducible K-module in the decomposition.

In [2], the author gave the branching rules for $SO(n+2)/SO(2)\times SO(n)$ $(n \geq 3)$ using a lemma on a type of determinant calculation. We shall study the branching rules for $SO(n+3)/SO(3)\times SO(n)$ $(n \geq 3)$, using the lemma again. We always consider $SO(k)$ as the matrix group consisting of real orthogonal $k\times k$ matrices with unit determinant.

2. WEYL'S CHARACTER FORMULA.

We shall review some basic facts on the representation of a Lie group.

When we have a group-homomorphism ρ from a Lie group G to the linear automorphism group of a finite-dimensional complex vector space V, we call the pair (V,ρ) a G-module. We sometimes omit to mention the homomorphism ρ, and say as a G-module V or V_G. A G-module is irreducible if and only if no non-trivial G-submodule under the same homomorphism ρ exists.

Let T be a maximal toral subgroup of a compact connected Lie group G. An irreducible G-module V_G decomposes into one-dimensional T-modules when the action of G through the homomorphism ρ is restricted to that of T. The action of T on each one-dimensional T-module is specified by an element λ, which is called the weight of the T-module, in a real vector subspace $\mathfrak{t}^w$ of the complexification of the dual vector space $\mathfrak{t}^*$ to the Lie algebra $\mathfrak{t}$ of T. The character χ_G of a G-module V_G is the formal sum of $\exp(\lambda)$ for all the one-dimensional T-modules in the decomposition. We may write it as $\chi_G=\sum_\lambda m_\lambda \exp(\lambda)$, putting the similar terms together. We fix a lexicographical ordering in the real vector space $\mathfrak{t}^w$. It is known that, in the weights of an irreducible G-module V_G, there exists a unique maximal weight Λ in this ordering, that is, the highest weight of V_G, which characterizes the irreducible G-module up to the G-isomorphisms. We denote the character of the G-module with the highest weight Λ by $\chi_G(\Lambda)$.

The Weyl group $W_G=N(T)/C(T)$, where $N(T)$ is the normalizer of T in G and $C(T)$ the centralizer of T in G, acts on $\mathfrak{t}$, on $\mathfrak{t}^*$, on its complexification, and on the weights of a G-module.

The complexification of the Lie algebra $\mathfrak{g}$ of G is a G-module under the Adjoint action. We call its weights the roots of G, and denote the set of all the roots by Δ_G. The set of all the positive roots, that is, the roots that are greater than 0 in the lexicographical ordering, are denoted by Δ_G^+. A positive root which cannot be a sum of two positive roots is called a simple root. It is known that W_G is generated by the mirror transformations with respect to the simple roots. If an element w is written as a product of an even number of mirror transformations with respect to simple roots, we give w the signature $+1$, otherwise the signature -1. We define the alternating character $\xi_G(\lambda)$ for an element λ in $\mathfrak{t}^w$ by

$$\xi_G(\lambda) = \sum_{w \in W_G} \mathrm{sgn}(w) \exp(w \cdot \lambda),$$

where $\mathrm{sgn}(w)$ is the signature of w.

We set $\delta_G = \frac{1}{2}\sum_{\alpha \in \Delta_G^+} \alpha$. Weyl's character formula gives us the means to calculate the character $\chi_G(\Lambda)$ from alternating characters:

$$\chi_G(\Lambda)\xi_G(\delta_G) = \xi_G(\Lambda + \delta_G).$$

Moreover, we know that $\xi_G(\delta_G)$ has the following expression.

$$\xi_G(\delta_G) = \prod_{\alpha \in \Delta_G^+} (\exp(\alpha/2) - \exp(-\alpha/2)).$$

In the following, we always take the maximal toral subgroup T of G so that T should include the maximal toral subgroup T' of K. Then, the character χ_K of the K-module that is obtained from a G-module V_G by the restriction of the action to K is nothing but the restriction $\chi_G|_{\mathfrak{t}'}$ of the character χ_G to the Lie algebra $\mathfrak{t}'$ of T'. When the irreducible G-module $V_G(\Lambda_G)$ with the highest weight Λ_G decomposes into the irreducible K-modules $V_K(\Lambda_K)$ with the highest weights Λ_K, we have the relation $\chi_G(\Lambda_G)|_{\mathfrak{t}'} = \sum_{\Lambda_K} \mathrm{mult}(\Lambda_K)\chi_K(\Lambda_K)$. Notice that $\mathrm{mult}(\Lambda_K)$ is the multiplicity of $V_K(\Lambda_K)$ in the decomposition of $V_G(\Lambda_G)$, that is, how many times a direct summand isomorphic to $V_K(\Lambda_K)$ appears in the decomposion.

Using Weyl's character formula for $\chi_G(\Lambda_G)$ and $\chi_K(\Lambda_K)$, we have

$$\sum_{\Lambda_K} \mathrm{mult}(\Lambda_K)\xi_K(\Lambda_K + \delta_K) = \left.\frac{\xi_G(\Lambda_G + \delta_G)}{\xi_G(\delta_G)}\right|_{\mathfrak{t}'} \cdot \xi_K(\delta_K).$$

Since the decomposition into the irreducible K-modules is unique, we can read out the branching rule once we can transform the right hand side to the sum of the form in the left hand side. We shall carry out this process for $SO(n+3)/SO(3) \times SO(n)$ in the next two sections.

3. Branching rules for $SO(2m+3)/SO(3) \times SO(2m)$.

We first treat the case $n = 2m$ $(m \geq 2)$. The subgroup $K = SO(3) \times SO(2m)$ of $G = SO(2m+3)$ consists of the elements $(a_{ij})_{1 \leq i,j \leq 2m+3}$ of G such that $(a_{ij})_{1 \leq i,j \leq 3} \in SO(3)$, $(a_{ij})_{4 \leq i,j \leq 2m+3} \in SO(2m)$, and other $a_{ij} = 0$. We take the maximal toral subgroup T of G as follows:

$$T=\left\{(a_{ij})\,\middle|\,\begin{array}{l} a_{11}=1,\\ \begin{pmatrix} a_{2k+2\ 2k+2} & a_{2k+2\ 2k+3}\\ a_{2k+3\ 2k+2} & a_{2k+3\ 2k+3}\end{pmatrix}\in SO(2)\quad (0\le k\le m),\\ \text{other } a_{ij}=0.\end{array}\right\}.$$

It is a direct product of $m+1$ $SO(2)$'s and is also the maximal toral subgroup of K.

Let $H=(a_{ij})$ be an element of the Lie algebra $\mathfrak{t}$ of T such that

$$\begin{pmatrix} a_{2k+2\ 2k+2} & a_{2k+2\ 2k+3}\\ a_{2k+3\ 2k+2} & a_{2k+3\ 2k+3}\end{pmatrix}=\begin{pmatrix} 0 & -X_k\\ X_k & 0\end{pmatrix}\quad (0\le k\le m),$$
$$\text{other } a_{ij}=0.$$

We define the elements $\lambda_0, \lambda_1, \ldots, \lambda_m$ of $\mathfrak{t}^w$ by $\lambda_k(H)=\sqrt{-1}X_k$ $(0\le k\le m)$. We fix a lexicographical ordering such that $\lambda_0\succ\lambda_1\succ\cdots\succ\lambda_m$. The positive roots Δ_G^+ of G and the positive roots Δ_K^+ of K are given by

$$\Delta_G^+=\left\{\begin{array}{ll}\lambda_k & (0\le k\le m),\\ \lambda_k+\lambda_\ell & (0\le k<\ell\le m),\\ \lambda_k-\lambda_\ell & (0\le k<\ell\le m).\end{array}\right\},$$

$$\Delta_K^+=\left\{\begin{array}{ll}\lambda_0, & \\ \lambda_k+\lambda_\ell & (1\le k<\ell\le m),\\ \lambda_k-\lambda_\ell & (1\le k<\ell\le m).\end{array}\right\}.$$

The hightest weight Λ_G of an irreducible G-module is of the form $\Lambda_G=h_0\lambda_0+h_1\lambda_1+\cdots+h_m\lambda_m$, where $h_0, h_1, \ldots, h_m$ are integers satisfying $h_0\ge h_1\ge\cdots\ge h_m\ge 0$. The highest weight Λ_K of an irreducible K-module is of the form $\Lambda_K=p_0\lambda_0+p_1\lambda_1+\cdots+p_{m-1}\lambda_{m-1}+\epsilon p_m\lambda_m$, where $p_0, p_1, \ldots, p_{m-1}, p_m$ are integers satisfying $p_0\ge 0$ and $p_1\ge\cdots\ge p_{m-1}\ge p_m\ge 0$, and ϵ is $+1$ or -1.

An irreducible G-module $V_G(\Lambda_G)$ is always the complexification of a real vector space with G-action. On the other hand, an irreducible K-module $V_K(\Lambda_K)$ is the complexification of an real vector space with K-action, when $p_m=0$. (In this case, ϵ is irrelevant.) The complexification of an real vector space with irreducible K-action is $V_K(\Lambda_K)$ with $p_m=0$ or a direct sum $V_K(\Lambda_K)+V_K(\overline{\Lambda_K})$ with $p_m>0$, where $\overline{\Lambda_K}$ is the Λ_K the sign ϵ of which is reversed. In the decomposition of $V_G(\Lambda_G)$, there appear only $V_K(\Lambda_K)$'s with $p_m=0$ or direct sums $V_K(\Lambda_K)+V_K(\overline{\Lambda_K})$ with $p_m>0$, since they must be the complexifications of real vector spaces with irreducible K-action. In this respect, we may restrict our attension to Λ_K with $\epsilon=1$ (or $p_m=0$), for, if $V_K(\Lambda_K)$ appears in the decomposition of $V_G(\Lambda_G)$, $V_K(\overline{\Lambda_K})$ also appears with the same multiplicity.

In the following, we set $s(\lambda)=\exp(\lambda)-\exp(-\lambda)$ and $c(\lambda)=\exp(\lambda)+\exp(-\lambda)$.

Theorem 1. *The irreducible K-module $V_K(\Lambda_K)$ with the highest weight $\Lambda_K=p_0\lambda_0+p_1\lambda_1+\cdots+p_m\lambda_m$ appears in the decomposition of the irreducible G-module $V_G(\Lambda_G)$ with the highest weight $\Lambda_G=h_0\lambda_0+h_1\lambda_1+\cdots+h_m\lambda_m$ if and only if the following conditions are satisfied.*

1. $p_m\le h_{m-1},\ p_{m-1}\le h_{m-2},\ h_{i+2}\le p_i\le h_{i-1}\ (1\le i\le m-2)$.

2. In the following expression, in which we calculate the left hand side and arrange them as in the right hand side, m_{p_0} *does not vanish:*

$$\sum_{(k_1,\dots,k_m)} \frac{\prod_{i=0}^{m} s(\ell_i \lambda_0)}{\left(s(\lambda_0)\right)^m} = \sum_{p \geq 0} m_p\, s\left(\left(p+\frac{1}{2}\right)\lambda_0\right),$$

where the sum in the left hand side is taken over all the sequences of integers $k_1, \dots, k_m$ *satisfying*

$$k_1 \geq \cdots \geq k_m \geq 0,$$
$$p_m \leq k_m \leq \min\{p_{m-1}, h_{m-1}\},$$
$$\max\{p_i, h_{i+1}\} \leq k_i \leq \min\{p_{i-1}, h_{i-1}\} \quad (2 \leq i \leq m-1),$$
$$\max\{p_1, h_2\} \leq k_1 \leq h_0,$$

and $\ell_0, \ell_1, \dots, \ell_m$ *are given by*

$$\ell_0 = h_0 - \max\{h_1, k_1\} + 1,$$
$$\ell_i = \min\{h_i, k_i\} - \max\{h_{i+1}, k_{i+1}\} + 1 \quad (1 \leq i \leq m-1),$$
$$\ell_m = \min\{h_m, k_m\} + \frac{1}{2}.$$

m_{p_0} *is the multiplicity of* $V_K(\Lambda_K)$ *in the decomposition.*

Proof. Adding up the positive roots, we know

$$\delta_G = \frac{2m+1}{2}\lambda_0 + \frac{2m-1}{2}\lambda_1 + \cdots + \frac{1}{2}\lambda_m,$$
$$\delta_K = \frac{1}{2}\lambda_0 + (m-1)\lambda_1 + \cdots + \lambda_{m-1}.$$

We set $\Lambda_G + \delta_G = \bar{h}_0\lambda_0 + \bar{h}_1\lambda_1 + \cdots + \bar{h}_m\lambda_m$. Then $\bar{h}_0, \bar{h}_1, \dots, \bar{h}_m$ are half integers, that is, real numbers of the form integers $+1/2$, satisfying $\bar{h}_0 > \bar{h}_1 > \cdots > \bar{h}_m > 0$. We also set $\Lambda_K + \delta_K = \bar{p}_0\lambda_0 + \bar{p}_1\lambda_1 + \cdots + \bar{p}_m\lambda_m$. Then $\bar{p}_0$ is a half integer satisfying $\bar{p}_0 > 0$ and $\bar{p}_1, \dots, \bar{p}_m$ are integers satisfying $\bar{p}_1 > \cdots > \bar{p}_m \geq 0$.

The alternating character $\xi_G(\Lambda_G + \delta_G)$ is the determinant of $(m+1)\times(m+1)$ matrix whose $(i+1, j+1)$-element is given by $s(\bar{h}_i\lambda_j)$:

$$\xi_G(\Lambda_G + \delta_G) = \det(s(\bar{h}_i\lambda_j))_{0 \leq i,j \leq m}.$$

Similarly we have

$$\xi_K(\Lambda_K + \delta_K) = s(\bar{p}_0\lambda_0) \times \left(\frac{1}{2}\right)\left(\det\left(c\left(\bar{p}_i\lambda_j\right)\right)_{1 \leq i,j \leq m} + \det\left(s\left(\bar{p}_i\lambda_j\right)\right)_{1 \leq i,j \leq m}\right).$$

If $\bar{p}_m = p_m = 0$, $\det(s(\bar{p}_i\lambda_j))$ vanishes. If $\bar{p}_m > 0$, we have

$$\xi_K(\Lambda_K + \delta_K) + \xi_K(\overline{\Lambda_K} + \delta_K) = s(\bar{p}_0) \times \det(c(\bar{p}_i\lambda_j))_{1 \leq i,j \leq m}.$$

It can be easily seen that

$$\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)} = \prod_{\alpha \in \Delta_G^+ \setminus \Delta_K^+} \left(\exp\left(\frac{\alpha}{2}\right) - \exp\left(-\frac{\alpha}{2}\right)\right)$$

$$= \prod_{j=1}^{m} \left[s\left(\frac{\lambda_j}{2}\right) s\left(\frac{\lambda_0 + \lambda_j}{2}\right) s\left(\frac{\lambda_0 - \lambda_j}{2}\right) \right].$$

What we should do is to expand the expression for $\xi_G(\Lambda_G + \delta_G)$ devided by $\xi_G(\delta_G)/\xi_K(\delta_K)$, that is,

$$\frac{\det\left(s\left(\bar{h}_i \lambda_j\right)\right)_{0 \le i,j \le m}}{\prod_{j=1}^{m} \left[s\left(\frac{\lambda_j}{2}\right) s\left(\frac{\lambda_0 + \lambda_j}{2}\right) s\left(\frac{\lambda_0 - \lambda_j}{2}\right) \right]}$$

in the linear combination of $s(\bar{p}_0)\det(c(\bar{p}_i \lambda_j))_{1 \le i,j \le m}$.

We shall use the following lemma in [2].

Lemma 2. *For half integers $\bar{h}_0, \bar{h}_1, \ldots, \bar{h}_m$ satisfying $\bar{h}_0 > \bar{h}_1 > \cdots > \bar{h}_m > 0$, we have*

$$\frac{\det\left(s\left(\bar{h}_i \lambda_j\right)\right)_{0 \le i,j \le m}}{\prod_{j=1}^{m} \left[s\left(\frac{\lambda_0 + \lambda_j}{2}\right) s\left(\frac{\lambda_0 - \lambda_j}{2}\right) \right]} = \sum_{(\bar{k}_1, \ldots, \bar{k}_m)} \frac{\prod_{i=0}^{m} s(\ell_i \lambda_0)}{\left(s(\lambda_0)\right)^m} \det\left(s\left(\bar{k}_i \lambda_j\right)\right)_{1 \le i,j \le m},$$

where the sum of the right hand side is taken over all the sequences of half integers $\bar{k}_1, \ldots, \bar{k}_m$ satisfying

$$\bar{k}_1 > \cdots > \bar{k}_m > 0,$$
$$\bar{k}_m < \bar{h}_{m-1}, \qquad \bar{h}_{i+1} < \bar{k}_i < \bar{h}_{i-1} \quad (1 \le i \le m-1),$$

and positive numbers $\ell_0, \ell_1, \ldots, \ell_m$ are given by

$$\ell_0 = h_0 - \max\{\bar{h}_1, \bar{k}_1\},$$
$$\ell_i = \min\{\bar{h}_i, \bar{k}_i\} - \max\{\bar{h}_{i+1}, \bar{k}_{i+1}\} \quad (1 \le i \le m-1),$$
$$\ell_m = \min\{\bar{h}_m, \bar{k}_m\}.$$

To get the theorem, it is enough to compute

$$\frac{\det\left(s\left(\bar{k}_i \lambda_j\right)\right)_{1 \le i,j \le m}}{\prod_{j=1}^{m} s\left(\frac{\lambda_j}{2}\right)}.$$

We have

$$\frac{s(\bar{k}_i \lambda_j)}{s(\lambda_j/2)} = 1 + c(\lambda_j) + c(2\lambda_j) + \cdots + c\left(\left(\bar{k}_i - \frac{1}{2}\right)\lambda_j\right).$$

Notice that $c(0\lambda_j) = 2$. Deviding each column of $\det\left(s(\bar{k}_i \lambda_j)\right)$ by $s(\lambda_j)$ and expanding it by the rows, we get

$$\frac{\det\left(s\left(\bar{k}_i \lambda_j\right)\right)_{1 \le i,j \le m}}{\prod_{j=1}^{m} s\left(\frac{\lambda_j}{2}\right)} = \sum_{(\bar{p}_1, \ldots, \bar{p}_m)} H \det\left(c\left(\bar{p}_i \lambda_j\right)\right)_{1 \le i,j \le m},$$

where the sum in the right hand side is taken over all the sequences of integers $\bar{p}_1, \ldots, \bar{p}_m$ satisfying

$$\bar{p}_1 > \cdots > \bar{p}_m \geq 0,$$
$$\bar{p}_m < \bar{k}_m, \qquad \bar{k}_{i+1} < \bar{p}_i < \bar{k}_i \quad (1 \leq i \leq m-1),$$

and $H = 1/2$ if $\bar{p}_m = 0$, $H = 1$ otherwise. We set $k_i = \bar{k}_i - (2m - 2i + 1)/2$ and $p_i = \bar{p}_i - (m - i)$ $(1 \leq i \leq m)$. Then, the integers $k_1, \ldots, k_m$ and $p_1, \ldots, p_m$ satisfy

$$k_1 \geq \cdots \geq k_m \geq 0, \qquad p_1 \geq \cdots \geq p_m \geq 0,$$
$$p_m \leq k_m, \qquad k_{i+1} \leq p_i \leq k_i \quad (1 \leq i \leq m-1),$$
$$k_m \leq h_{m-1}, \qquad h_{i+1} \leq k_i \leq h_{i-1} \quad (1 \leq i \leq m-1).$$

Combining these inequalities, we get the condition for $\det(c(\bar{p}_i \lambda_j))$ multiplied by a linear combination of $s(p\lambda_0)$ to be included in the expansion of $\xi_G(\Lambda_G + \delta_G)$ devided by $\xi_G(\delta_G)/\xi_K(\delta_K)$. It is easy to see that the linear combination of $s(p\lambda_0)$ is what is given in the theorem. □

Example 3. By the theory of spherical functions, we know that an irreducible G-module $V_G(\Lambda_G)$ with the highest weight Λ_G includes the trivial K-module $\mathbf{C}$ in the decomposition if and only if Λ_G is a linear combination of the so-called fundamental weights of the pair (G, K) with non-negative integer coefficients. Moreover the multiplicity is always one. We check this by our branching rule.

As the highest weight Λ_K of the trivial K-module $\mathbf{C}$ is 0, we set $p_0 = p_1 = \cdots = p_m = 0$. If $\mathbf{C}$ is included in the decomposition of $V_G(\Lambda_G)$, we have $h_3 = \cdots = h_m = 0$. The condition on the sequence of integers $k_1, \ldots, k_m$ is given by $h_2 \leq k_1 \leq h_0$ and $k_2 = \cdots = k_m = 0$, and we set $\ell_0 = h_0 - \max\{h_1, k_1\} + 1$, $\ell_1 = \min\{h_1, k_1\} - h_2 + 1$, $\ell_2 = \cdots = \ell_{m-1} = 1$, $\ell_m = 1/2$. We shall compute the expansion of

$$\frac{s(\lambda_0/2)}{\left(s(\lambda_0)\right)^2} \sum_{k_1 = h_2}^{h_0} \left(s(\ell_0 \lambda_0) s(\ell_1 \lambda_0)\right)$$

and see what is the coefficient of $s(\lambda_0/2)$.

Using $s(\ell\lambda)/s(\lambda) = \sum_{i=0}^{\ell-1} \exp\left((\ell - 1 - 2i)\lambda\right)$,

we have

$$\begin{aligned}
&\frac{s(\ell_0\lambda_0)}{s(\lambda_0)} \frac{s(\ell_1\lambda_0)}{s(\lambda_0)} \\
&\quad = \sum_{i=0}^{\ell_0 - 1} \sum_{j=0}^{\ell_1 - 1} \exp\left((\ell_0 - 1 - 2i)\lambda_0\right) \exp\left((\ell_1 - 1 - 2j)\lambda_0\right) \\
&\quad = \sum_{i=0}^{\min\{\ell_0, \ell_1\} - 2} (i+1) \exp\left((\ell_0 + \ell_1 - 2 - 2i)\lambda_0\right) \\
&\qquad + \sum_{i = \min\{\ell_0, \ell_1\} - 1}^{\max\{\ell_0, \ell_1\} - 1} \min\{\ell_0, \ell_1\} \exp\left((\ell_0 + \ell_1 - 2 - 2i)\lambda_0\right) \\
&\qquad + \sum_{i = \max\{\ell_0, \ell_1\}}^{\ell_0 + \ell_1 - 2} (\ell_0 + \ell_1 - i - 1) \exp\left((\ell_0 + \ell_1 - 2 - 2i)\lambda_0\right) \\
&\quad = \begin{cases} \min\{\ell_0, \ell_1\} \cdot 1 + \sum_{q \geq 1} c_{2q}\, c(2q\lambda_0) & \ell_0 + \ell_1 : \text{even}, \\ \min\{\ell_0, \ell_1\} \cdot c(\lambda_0) + \sum_{q \geq 1} c_{2q+1}\, c\left((2q+1)\lambda_0\right) & \ell_0 + \ell_1 : \text{odd}, \end{cases}
\end{aligned}$$

where c_{2q} or c_{2q+1} are non-negative integers. Since we have $s(\lambda/2)c(\ell\lambda)=s((\ell+1/2)\lambda)-s((\ell-1/2)\lambda)$, the coefficient of $s((1/2)\lambda_0)$ in question is given by

$$\sum_{k_1=h_2}^{h_0}(-1)^{\ell_0+\ell_1}\min\{\ell_0,\ell_1\}=\begin{cases}1, \text{ if both } h_1-h_2 \text{ and } h_0-h_1 \text{ are even,}\\ 0, \text{ otherwise.}\end{cases}$$

Therefore, an irreducible G-module $V_G(\Lambda_G)$ contains a trivial K-module $\mathbf{C}$ in the irreducible decomposition as a K-module, if and only if Λ_G is a linear combination of $2\lambda_0$, $2(\lambda_0+\lambda_1)$, and $\lambda_0+\lambda_1+\lambda_2$ with non-negative integral coefficients, and, then, the multiplicity is always 1. □

4. Branching rules for $SO(2m+4)/SO(3)\times SO(2m+1)$.

We next treat the case $n=2m+1$ $(m\geq 1)$. The subgroup $K=SO(3)\times SO(2m+1)$ of $G=SO(2m+4)$ consists of the elements $(a_{ij})_{1\leq i,j\leq 2m+4}$ of G such that $(a_{ij})_{1\leq i,j\leq 3}\in SO(3)$, $(a_{ij})_{4\leq i,j\leq 2m+4}\in SO(2m+1)$, and other $a_{ij}=0$. We take the maximal toral subgroup T of G as follows:

$$T=\left\{(a_{ij})\left|\begin{array}{l}\begin{pmatrix}a_{2k+1\ 2k+1} & a_{2k+1\ 2k+2}\\ a_{2k+2\ 2k+1} & a_{2k+2\ 2k+2}\end{pmatrix}\in SO(2)\quad(0\leq k\leq m+1),\\ \text{other } a_{ij}=0.\end{array}\right.\right\}.$$

It is a direct product of $m+2$ $SO(2)$'s. The maximal toral subgroup T' of K that is included in T is given by

$$T'=\left\{(a_{ij})\in T\left|\begin{pmatrix}a_{33} & a_{34}\\ a_{43} & a_{44}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right.\right\}.$$

Let $H=(a_{ij})$ be an element of the Lie algebra $\mathfrak{t}$ of T such that

$$\begin{pmatrix}a_{2k+3\ 2k+3} & a_{2k+3\ 2k+4}\\ a_{2k+4\ 2k+4} & a_{2k+4\ 2k+4}\end{pmatrix}=\begin{pmatrix}0 & -X_k\\ X_k & 0\end{pmatrix}\quad(-1\leq k\leq m),$$
$$\text{other } a_{ij}=0.$$

We define the elements λ_{-1}, λ_0, λ_1, …, λ_m of $\mathfrak{t}^*$ by $\lambda_k(H)=\sqrt{-1}X_k$ $(-1\leq k\leq m)$. We fix a lexicographical ordering such that $\lambda_{-1}\succ\lambda_0\succ\lambda_1\succ\cdots\succ\lambda_m$. The positive roots Δ_G^+ of G and the positive roots Δ_K^+ of K are given by

$$\Delta_G^+=\left\{\begin{array}{l}\lambda_k+\lambda_\ell\quad(-1\leq k<\ell\leq m),\\ \lambda_k-\lambda_\ell\quad(-1\leq k<\ell\leq m).\end{array}\right\},$$

$$\Delta_K^+=\left\{\begin{array}{l}\lambda_{-1},\ \ \lambda_k\quad(1\leq k\leq m),\\ \lambda_k+\lambda_\ell\quad(1\leq k<\ell\leq m),\\ \lambda_k-\lambda_\ell\quad(1\leq k<\ell\leq m).\end{array}\right\}.$$

Notice that the Lie algebra $\mathfrak{t}'$ of T' is the subspace of $\mathfrak{t}$ defined by $\lambda_0=0$.

The hightest weight Λ_G of an irreducible G-module is of the form $\Lambda_G=h_{-1}\lambda_{-1}+h_0\lambda_0+h_1\lambda_1+\cdots+h_{m-1}\lambda_{m-1}+\epsilon h_m\lambda_m$, where h_{-1}, h_0, h_1, …, h_{m-1}, h_m are integers satisfying $h_{-1}\geq h_0\geq h_1\geq\cdots\geq h_{m-1}\geq h_m\geq 0$ and ϵ is $+1$ or -1. The highest weight Λ_K of an irreducible K-module is of the form $\Lambda_K = p_{-1}\lambda_{-1} + p_1\lambda_1 + \cdots + p_m\lambda_m$, where p_{-1}, p_1, ..., p_m are integers satisfying $p_{-1} \geq 0$ and $p_1 \geq \cdots \geq p_m \geq 0$.

Theorem 4. *The irreducible K-module $V_K(\Lambda_K)$ with the highest weight $\Lambda_K = p_{-1}\lambda_{-1} + p_1\Lambda_1 + \cdots + p_m\lambda_m$ appears in the decomposition of the irreducible G-module $V_G(\Lambda_G)$ with the highest weight $\Lambda_G = h_{-1}\lambda_{-1} + h_0\lambda_0 + h_1\lambda_1 + \cdots + h_{m-1}\lambda_{m-1} + \epsilon h_m\lambda_m$ if and only if the following conditions are satisfied.*

1. $p_m \leq h_{m-2}$, $h_{i+1} \leq p_i \leq h_{i-2}$ $(1 \leq i \leq m-1)$.

2. In the following expression, in which we calculate the left hand side and arrange them as in the right hand side, $m_{p_{-1}}$ does not vanish:

$$\sum_{(q_0, q_1, \dots, q_m)} \frac{\prod_{i=0}^{m} s(r_i\lambda_{-1})}{(s(\lambda_{-1}))^m} = \sum_{p \geq 0} m_p\, s\left(\left(p + \frac{1}{2}\right)\lambda_{-1}\right),$$

where the sum in the left hand side is taken over all the sequences of integers q_0, q_1, ..., q_m satisfying

$$q_0 \geq q_1 \geq \cdots \geq q_m \geq 0,$$
$$h_m \leq q_m \leq \min\{p_{m-1}, h_{m-1}\},$$
$$\max\{p_{i+1}, h_i\} \leq q_i \leq \min\{p_{i-1}, h_{i-1}\} \quad (2 \leq i \leq m-1),$$
$$\max\{p_2, h_1\} \leq q_1 \leq h_0, \qquad \max\{p_1, h_0\} \leq q_0 \leq h_{-1},$$

and r_0, r_1, ..., r_m are given by

$$r_0 = q_0 - \max\{q_1, p_1\} + 1,$$
$$r_i = \min\{q_i, p_i\} - \max\{q_{i+1}, p_{i+1}\} + 1 \quad (1 \leq i \leq m-1),$$
$$r_m = \min\{q_m, p_m\} + \frac{1}{2}.$$

$m_{p_{-1}}$ is the multiplicity of $V_K(\Lambda_K)$ in the decomposition.

Proof. Adding up the positive roots, we know

$$\delta_G = (m+1)\lambda_{-1} + m\lambda_0 + (m-1)\lambda_1 + \cdots + \lambda_{m-1},$$
$$\delta_K = \frac{1}{2}\lambda_{-1} + \frac{2m-1}{2}\lambda_1 + \cdots + \frac{1}{2}\lambda_m.$$

We set $\Lambda_G + \delta_G = \bar{h}_{-1}\lambda_{-1} + \bar{h}_0\lambda_0 + \bar{h}_1\lambda_1 + \cdots + \bar{h}_{m-1}\lambda_{m-1} + \epsilon\bar{h}_m\lambda_m$. Then $\bar{h}_0$, $\bar{h}_1$, ..., $\bar{h}_m$ are integers satisfying $\bar{h}_{-1} > \bar{h}_0 > \bar{h}_1 > \cdots > \bar{h}_{m-1} > \bar{h}_m \geq 0$. We also set $\Lambda_K + \delta_K = \bar{p}_{-1}\lambda_{-1} + \bar{p}_1\lambda_1 + \cdots + \bar{p}_m\lambda_m$. Then $\bar{p}_{-1}$, $\bar{p}_1$, ..., $\bar{p}_m$ are half integers satisfying $\bar{p}_{-1} > 0$ and $\bar{p}_1 > \cdots > \bar{p}_m > 0$.

The alternating characters $\xi_G(\Lambda_G + \delta_G)$ and $\xi_K(\Lambda_K + \delta_K)$ are given by

$$\xi_G(\Lambda_G + \delta_G) = \frac{1}{2}\left(\det\left(c(\bar{h}_i\lambda_j)\right)_{-1 \leq i,j \leq m} + \epsilon\det\left(s(\bar{h}_i\lambda_j)\right)_{-1 \leq i,j \leq m}\right),$$
$$\xi_K(\Lambda_K + \delta_K) = s(\bar{p}_{-1}\lambda_{-1}) \times \det\left(s(\bar{p}_i\lambda_j)\right)_{1 \leq i,j \leq m}.$$

By setting $\lambda_0 = 0$, we get the restriction of $\xi_G(\Lambda_G + \delta_G)$ to $\mathfrak{t}'$:

$$\xi_G(\Lambda_G+\delta_G)\big|_{\mathfrak{t}'} = \det\begin{pmatrix} c(\bar{h}_{-1}\lambda_{-1}) & 1 & c(\bar{h}_{-1}\lambda_1) & \cdots & c(\bar{h}_{-1}\lambda_m) \\ c(\bar{h}_0\lambda_{-1}) & 1 & c(\bar{h}_0\lambda_1) & \cdots & c(\bar{h}_0\lambda_m) \\ c(\bar{h}_1\lambda_{-1}) & 1 & c(\bar{h}_1\lambda_1) & \cdots & c(\bar{h}_1\lambda_m) \\ \vdots & \vdots & \vdots & & \vdots \\ c(\bar{h}_m\lambda_{-1}) & 1 & c(\bar{h}_m\lambda_1) & \cdots & c(\bar{h}_m\lambda_m) \end{pmatrix}$$

$$= (-1)^m \det\big(c(\bar{h}_{i-1}\lambda_j) - c(\bar{h}_i\lambda_j)\big)_{\substack{0\le i\le m \\ j=-1 \text{ or } 1\le j\le m}}.$$

In the same way, we have

$$\frac{\xi_G(\delta_G)\big|_{\mathfrak{t}'}}{\xi_K(\delta_K)} = (-1)^m s\left(\frac{\lambda_{-1}}{2}\right)\prod_{j=1}^{m}\left(s\left(\frac{\lambda_j}{2}\right)s\left(\frac{\lambda_{-1}+\lambda_j}{2}\right)s\left(\frac{\lambda_{-1}-\lambda_j}{2}\right)\right).$$

Applying the formula

$$\frac{c(\bar{h}_{i-1}\lambda) - c(\bar{h}_i\lambda)}{s(\lambda/2)} = \sum_{\bar{q}=\bar{h}_i+1/2}^{\bar{h}_{i-1}-1/2} s(\bar{q}\lambda),$$

on each column of the matrix in $\xi_G(\Lambda_G+\delta_G)\big|_{\mathfrak{t}'}$, and expanding it by the rows, we obtain

$$\left.\frac{\xi_G(\Lambda_G+\delta_G)}{\xi_G(\delta_G)}\right|_{\mathfrak{t}'}\cdot\xi_K(\delta_K) = \sum_{(\bar{q}_0,\bar{q}_1,\dots,\bar{q}_m)} \frac{\det(s(\bar{q}_i\lambda_j))_{\substack{0\le i\le m \\ j=-1 \text{ or } 1\le j\le m}}}{\prod_{j=1}^{m}\left(s\left(\frac{\lambda_{-1}+\lambda_j}{2}\right)s\left(\frac{\lambda_{-1}-\lambda_j}{2}\right)\right)},$$

where the sum in the right hand side is taken over all the sequences of half integers $\bar{q}_0, \bar{q}_1, \dots, \bar{q}_m$ satisfying $\bar{h}_{-1} > \bar{q}_0 > \bar{h}_0 > \bar{q}_1 > \bar{h}_1 > \dots > \bar{h}_{m-1} > \bar{q}_m > \bar{h}_m \ge 0$. We set $q_j = \bar{q}_j - (2(m-j)+1)/2$ $(0 \le j \le m)$. In view of Lemma 2, we can easily deduce Theorem 4 from this formula. □

Example 5. We again consider which irreducible G-module $V_G(\Lambda_G)$ with the highest weight Λ_G includes the trivial K-module $\mathbf{C}$ in the decomposition. As in Example 3, we set $p_{-1} = 0$ and $p_1 = \dots = p_m = 0$, and, then, we have $h_2 = \dots = h_m = 0$. The condition on the sequences of integers q_0, $q_1, \dots, q_m$ are given by $h_{-1} \ge q_0 \ge h_0 \ge q_1 \ge h_1$ and $q_2 = \dots = q_m = 0$. We have $r_0 = q_0 - q_1 + 1$, $r_1 = \dots = r_{m-1} = 1$, and $r_m = 1/2$. We shall compute the expansion of

$$s\left(\frac{\lambda_{-1}}{2}\right)\sum_{q_1=h_1}^{h_0}\sum_{q_0=h_0}^{h_{-1}}\frac{s((q_0-q_1+1)\lambda_{-1})}{s(\lambda_{-1})}$$

and see what is the coefficient of $s(\lambda_{-1}/2)$. Since we have

$$\frac{s((q_0-q_1+1)\lambda_{-1})}{s(\lambda_{-1})} = \begin{cases} 1 + \displaystyle\sum_{q=1}^{(q_0-q_1)/2} c(2q\lambda_{-1}) & q_0 - q_1 : \text{even}, \\ c(\lambda_{-1}) + \displaystyle\sum_{q=1}^{(q_0-q_1-1)/2} c((2q+1)\lambda_{-1}) & q_0 - q_1 : \text{odd}, \end{cases}$$

the coefficient is equal to the number of the pairs (q_0, q_1) satisfying $h_{-1} \geq q_0 \geq h_0 \geq q_1 \geq h_1$ such that $q_0 - q_1$ is even minus the number of the pairs (q_0, q_1) satisfying $h_{-1} \geq q_0 \geq h_0 \geq q_1 \geq h_1$ such that $q_0 - q_1$ is odd. The number is 1 if both $h_{-1} - h_0$ and $h_0 - h_1$ are even, and 0 otherwise.

Therefore, when $m > 1$, an irreducible G-module $V_G(\Lambda_G)$ contains a trivial K-module $\mathbf{C}$ in the irreducible decomposition as a K-module, if and only if Λ_G is a linear combination of $2\lambda_{-1}$, $2(\lambda_{-1} + \lambda_0)$, and $\lambda_{-1} + \lambda_0 + \lambda_1$ with non-negative integral coefficients, and, then, the multiplicity is always 1. When $m = 1$, an irreducible G-module $V_G(\Lambda_G)$ contains a trivial K-module $\mathbf{C}$ in the irreducible decomposition as a K-module, if and only if Λ_G is a linear combination of $2\lambda_{-1}$, $\lambda_{-1} + \lambda_0 + \lambda_1$, and $\lambda_{-1} + \lambda_0 - \lambda_1$ with non-negative integral coefficients, and, then, the multiplicity is always 1. □

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